PLANE
GEOMETRY

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CORRELATED MATHEMATICS FOR SECONDARY SCHOOLS

PLANE GEOMETRY

BY

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PREFACE

In this text, as in the "Algebra," the authors have attempted to present the subject-matter in a way more suitable to beginners than is the case in most of the modern books. Brief and concise methods of exposition are therefore largely avoided, especially at the start, and much space is given to the explanation and analysis of theorems and problems, in order to bring out clearly the plan of attack and the method of proof.

In the presentation of theorems, where complete proofs are given, the following plan has been adopted:

1. Statement of theorem.
2. Figure.
3. Statement of what is given. (Hypothesis.)
4. Statement of what is to be proved. (Conclusion.)
5. Analysis.
6. Proof.

Special attention is called to the analyses. If the student can be taught to analyze a problem clearly, he has taken a long step in its solution. It is at this point that the logical faculties, and the ability to co-ordinate and apply information previously acquired, receive their chief development, and too much emphasis can hardly be placed on this point. The proof then consists merely in establishing the steps marked out in the analysis.

In a number of theorems and problems, only the statement and analysis are given; in others the statement alone. These should be assigned as exercises, the student being required to work out complete proofs and to file them in a carefully kept note-book.

There is no better way to vivify the treatment of Algebra than through geometric interpretation of its magnitudes; likewise, Geometry can, and does, borrow much from Algebra in using its rules and symbols in the study of geometric forms. The authors wish to emphasize the fact that in this text the simpler operations of Algebra are brought often into play, so that the student is made to utilize the facts acquired in his first years' study of mathematics to aid him in the work of the second year. Important illustrations of this will be found in the geometric treatment of factoring, and in the presentation of the subject of ratio and proportion, which is developed both algebraically and geometrically.

The lists of exercises will be found very complete, and to contain numerous illustrations of the practical aspects of Geometry. No attempt should be made to cover them all in one year, but rather to select and choose a variety, sufficient to give practice and to illustrate the meaning and bearing of the text.

There is perhaps nothing which mars the usefulness of the study of Geometry so much as the habit of making careless and slipshod constructions. The teacher is urged to insist on the use of a sharp pencil, not too soft, and of rule and compass, for the latter a metal point attached to the pencil if nothing better is available, and to have drawings made to exact scale and measured, whenever possible. Much work of this sort is called for in the exercises.

Some of the results of careless drawing are shown in the paradoxes at the end of the book. It will be well to present these to the class when suitable places are reached in the text.

The discussion of incommensurable cases also has been placed at the end of the book, after some study of the notion of limits. They may be taken up at proper points in the text, or, better, at the end of the course.

References to the authors' text-book on Algebra are made under the designation "First Course." Most of the subject-matter referred to may be found in any other modern text.

E. Long,
W. C. Brenke.

Lincoln, Nebraska,
May, 1916.

CONTENTS

SYMBOLS

The symbols $+$, $-$, $\times$, $\div$, are used as in Algebra.

$=$ equals, is equal to.

$\neq$ is not equal to.

$<$ is less than.

$>$ is greater than.

$\backsim$ similar, is similar to.

$\cong$ similar and equal, i. e. congruent.

$\parallel$ parallel, is parallel to; $\parallel$s parallels.

$\perp$ perpendicular, is perpendicular to; $\perp$s perpendiculars.

$\angle$, $\angle$s angle, angles.

rt. $\angle$ right angle.

$\triangle$, $\triangle$s triangle, triangles.

▱, ▱s parallelogram, parallelograms.

▭, ▭s rectangle, rectangles.

○, ⊙s circle, circles.

⌒ arc.

$\therefore$ therefore.

$\doteq$ approaches.

PLANE GEOMETRY

CHAPTER I

PART I—HISTORICAL INTRODUCTION. PART II—GEOMETRIC FORMS. POSTULATES OF THE STRAIGHT LINE. AXIOMS.

PART I—HISTORICAL INTRODUCTION

Before beginning the study of geometry it will be well for the student to make a brief survey of the history of the development of the subject, that he may approach it with his mind open to grasp its twofold nature, the practical and the logical. Many of the truths stated in the geometry are of the greatest value to the constructive work of the world, while, from the educational standpoint, the proof that the statements made are true trains the mind in clear logical reasoning.

On the practical side, the pyramids and temples of Egypt tell us that many of the truths of geometry were known and used by builders thousands of years before Christ, while a papyrus written by Ahmes (now preserved in the British museum) records the mathematics known to the Egyptians as far back as 2500 B. C., but gives no intimation that the people made any attempt to demonstrate the general truth of geometrical theorems.

To the Egyptian geometry was of practical value. It remained for the Greeks to give to the world its pure logical reasoning.

Thales of Miletus lived from 640 to 546 B. C. Being a merchant, his commercial pursuits took him into Egypt, where the great buildings attracted his attention. They had attracted the attention of others before him but not in the same way. These were the questions that bothered Thales: How was it possible for the builders to place them on a direct north and south line? How did they make one side of the base at exact right angles to the other? How did they build into the air with such perfect slope and symmetry? He went to Egypt as a merchant; he remained as a student. He studied with the priests of the country and soon excelled them in knowledge. He surprised them when by a simple device he measured the heights of the pyramids.

When he had learned all that he could, he returned to Greece where he gathered about him scholarly men whom he interested in the subject, and formed a sort of club to pursue its study. This group of scholars is known in history as the Ionian school. They did not confine their attention to mathematics but also studied philosophy and astronomy. In mathematics they were the first to study lines, angles, and surfaces, apart from the solids on which they found them. The truths they proved were, naturally, of the simplest nature.

Thales is known as one of the "seven wise men of Greece." He died about 546 B. C.

Some years before the death of Thales another famous school was started, known as the Pythagorean school. The leader of this school was **Pythagoras,** born in Samos, 580 B. C. The history of his early life is not clearly known but it is thought that he visited Thales and his school and upon advice given there went to Egypt and Babylon to study.

He then returned to Greece where he tried to establish a school at Samos, his birthplace. This was not a success, so he went to Croton, Italy. Here he met with great success.

The school accomplished many great things, the greatest of which was to make the study of geometry a form of liberal education. The plan of this school differed greatly from that of the Ionian school. It was very secret. No one was allowed to divulge anything that went on. Especially was it offensive if any one boasted of his own achievements to those outside. Pythagoras took all credit to himself, although it is now known that there were many brilliant minds connected with the work. This mystery was the cause of an attack made upon the school, resulting in the destruction of the buildings. Pythagoras fled, but was pursued and killed. The school itself lasted for about 200 years, although it did not continue as a secret brotherhood.

Again we find an overlapping of schools. Before the Pythagoreans had entirely disappeared, there arose in Athens a group of men known as the Sophists, "wise men" who taught the people rhetoric, philosophy, and mathematics. This was more like our present day school, since the leaders took pay for their work. The Pythagoreans scorned to take money for imparting knowledge.

Somewhat later during the time that Athens held high place in the literary world, **Plato** founded his school known as the Platonic school. Plato was a philosopher and not a professed mathematician, but he took hold of the study and brought to it carefully stated definitions, analysis, and logical methods of reasoning. He stimulated the study by placing over the entrance of his school, "Let no one who is unacquainted with geometry enter here." Athens was conquered by Philip of Macedon and her great power broken in 338 B. C.

Shortly after the destruction of Athens our history takes us back to Egypt. Alexander the Great founded Alexandria, which was shortly to become the seat of the greatest university of the time, known as the First Alexandrian School. One of its first and most noted teachers of mathematics was

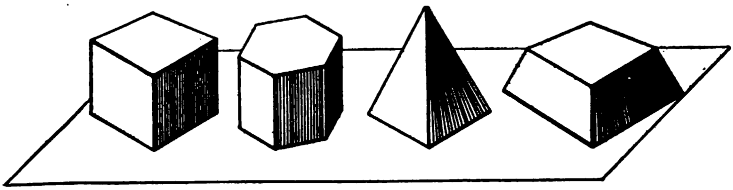
Euclid. Let the student fix this name well in mind for it was Euclid who wrote our first geometry. He lived about 300 B. C. It has been stated that no ancient writer in any branch of knowledge has held such a commanding position in modern education as has Euclid in his "Elements of Geometry." He gathered the work of the preceding centuries, completed proofs, added new proofs, and arranged the whole in logical order. The modern texts, such as you are about to study, are largely selected from this work of Euclid. The original work consisted of thirteen chapters usually spoken of as books.

A man who has given us much in mathematics was a student in the Alexandrian school. This man was **Archimedes** (287–212 B. C.). You will meet with his name often in physics, since he excelled in the discoveries of physical laws. However, your attention will be called especially to him when you take up the study of solid geometry.

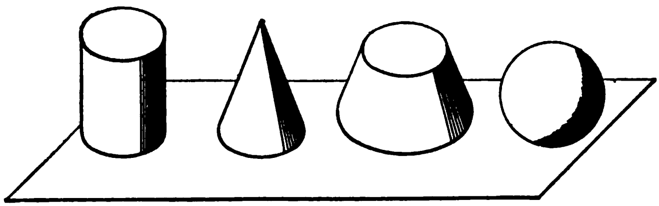
This short account is but the setting of the history of geometry. As the work develops, attention will be called to the work done in the individual schools as far as it is known. From this we hope that you may see how geometry began in Egypt, practical but without logical reasoning, was carried to the Ionian Isles and nourished, grew rapidly in the Pythagorean school in Italy, received system and logic from Plato in Athens, and was carried back to Egypt to be brought into a state of high perfection by Euclid.

PART II—GEOMETRIC FORMS. POSTULATES OF THE STRAIGHT LINE. AXIOMS.

1. As you go about out of doors notice the forms of objects, such as the roofs of houses, spires, doors and windows of churches, trees and shrubs. Make groups of the names of these objects as in your opinion they conform in a general way to one of the following forms.

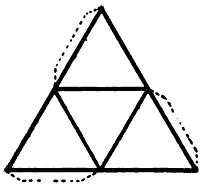


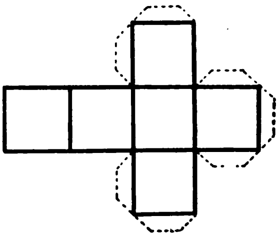
Cube Prism Pyramid Frustum of Pyramid

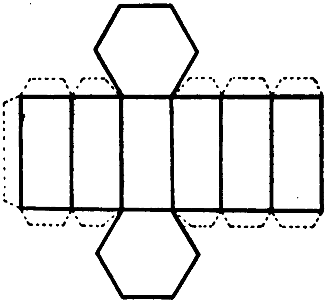


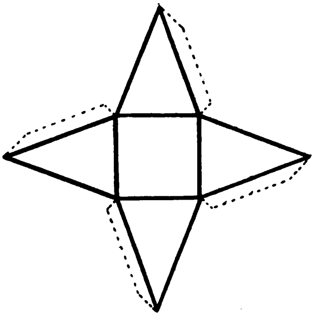
Cylinder Cone Frustum of Cone Sphere

2. It is easy to make models of some of these forms. From stiff paper or cardboard cut out at least one of the following patterns, which should be enlarged four or five times; fold along the heavy lines and use the flaps marked by dotted lines to fasten the figure together. The drawing of the enlarged pattern will require some assistance from the instructor. If desired, the flaps may be left off and the model fastened together with gummed paper strips.









The student should also make some pattern of his own design.

3. Suppose that you had made your model of tin instead of paper so that you could fill it with water, salt, mercury, what change would take place as you filled it with each in turn? If it were air tight and you should pump out all the air, how would your model differ from what it was before?

4. When your model is filled with air, is not the inner air as completely separated from the outer air as in the other cases, the water, salt, or mercury is separated from the outer air? The weight of your model is changed but there is one thing that is not changed, and that is the shape. Now if you can imagine the cardboard out of which your model is made to become thinner and thinner until it is so thin that you cannot see it at all, still keeping in mind a body of definite shape, separated from everything else, you have in mind what is called a **geometric solid.** That which you think of as separating this geometric solid from everything about it is called a **geometric surface.**

5. However, your figure as it is, is a geometric solid, provided you agree to disregard everything about it but shape and size. Thus all objects that you see are solids as well as shapes that you have in mind and cannot see. The cardboard used to make your pattern is solid and is separated from the air about it by surface which is neither air nor cardboard.

6. Examine your model; you will see that the surface is divided into different portions. That which divides one part of surface from another is called a **line.** Are there lines on your model? Point out the lines. Again these lines are divided into different portions by means of **points.** Are there points on your model? Are these lines and points, air, cardboard, or neither?

7. We shall call the surfaces of the solids **faces.** The lines where the faces intersect are called **edges,** and the point where the edges intersect are called **vertices** (corners).

Count the number of faces, the number of edges and the number of vertices of each of your models, and make a table of the numbers.

Faces	Vertices	Edges	$f + v = e + 2$
6	8	12	$6 + 8 = 12 + 2$

Do you find that the same relation exists between the numbers in each case? If you let f stand for the number of faces, and v for the number of vertices, and e for the number of edges, you should find this to be true:

$$f + v = e + 2.$$

8. It was stated above that that which divides one portion of surface from another is called a line. Can the surface of your model be divided by other lines than edges? In how many different ways may it be divided by different lines? In other words, how many lines are there on the surface of your model? Think out this question with reference to the invisible block that we considered. Can these lines be divided by means of points other than vertices? How many points are there on the surface of your model?

9. A geometric point has neither length, breadth, nor thickness. When you place the "point" of your pencil on a paper so as to make a visible mark, have you made a geometric point? Is the statement true that between two points there can always be placed an unlimited number of other points, no matter how close together the points are chosen?

10. Can you think of a moving point? If so, moving in how many different directions? How far? What would you suggest as the name of the path? If you think of two fixed points, in how many different ways could a third point pass from one to the other? Draw a picture of some of the lines generated by a moving point passing from one fixed point to another. Of these lines the one which brings the same idea to our minds as soon as its name is mentioned is called a **straight line.** How many such lines can be drawn between two points? What are some of the properties of a straight line? Which of these properties do the two points determine?

In how many points can two straight lines intersect?

The above discussion brings out four facts whose truth we assume, and which will be needed in the work before us. They are called **postulates** for the straight line.

Post. I. *Between two points only one straight line can be drawn.*

Post. II. *Two points determine a straight line.*

Post. III. *The shortest distance between two points is measured on the straight line joining them.*

Post. IV. *Two intersecting lines determine a point.*

In geometry, as in algebra, we shall also use the following assumptions, called **axioms.**

Ax. 1. *Things equal to the same thing are equal to each other.*

Ax. 2. *If equals are added to equals, the sums are equal.*

Ax. 3. *If equals are subtracted from equals, the remainders are equal.*

Ax. 4. *If equals are multiplied by equals, the products are equal.*

Ax. 5. *If equals are divided by equals, the quotients are equal.*

Ax. 6. *If equals are added to unequals, the sums are unequal in the same order.*

Ax. 7. *If equals are subtracted from unequals, the remainders are unequal in the same order.*

Ax. 8. *If unequals are multiplied by positive equals, the products are unequal in the same order.*

Ax. 9. *If unequals are divided by positive equals, the quotients are unequal in the same order.*

Ax. 10. *If unequals are added to unequals, greater to greater and less to less, the sums are unequal in the same order.*

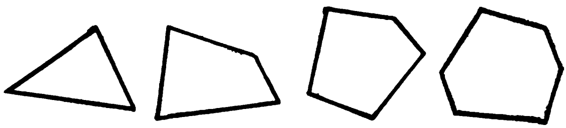
Ax. 11. *Like powers of equals, or like roots, are equal.*

Ax. 12. *If three magnitudes are such that the first is greater than the second and the second greater than the third, then the first is greater than the third.*

Ax. 13. *The whole is greater than any of its parts.*

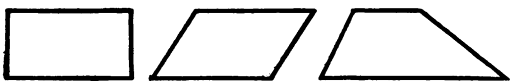
Ax. 14. *The whole is equal to the sum of all its parts.*

11. Examining various geometric solids you will find that some have flat surfaces and others have curved surfaces, while some have both. Those which have curved surfaces such as the cylinder, cone, sphere, are frequently referred to as the round bodies, while those with all surfaces flat are called polyhedra. You will find that the plane surfaces are of various shapes of which the following are the most common:



Triangle Quadrilateral Pentagon Hexagon

Among the quadrilaterals you will find the following:



Rectangle Parallelogram Trapezoid

In your homes note the designs of wall-paper, linoleum, rugs and tiling. You will find many interesting geometric shapes among them.

REVIEW OF CHAPTER I

1. Geometry deals primarily with form. Name the common forms. Do things that grow take these shapes? What parts of buildings have these various forms?

2. Surfaces are either curved or flat. Name solids that have only flat surfaces. Name solids that have both curved and flat surfaces. Is there any that has curved surface only?

3. What forms of faces are found on the solids that have flat surfaces only? What forms of flat faces are found on those that have both flat and curved surfaces?

4. The objects of geometric study are solids, surfaces, lines and points.

5. State the postulates for the straight line.

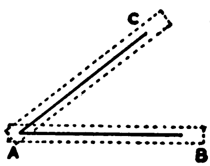
6. State the axioms in algebraic language, as, for example, Ax. 6: If $a = b$ and $c < d$, then $a + c < b + d$.

CHAPTER II

PART I—ANGLES. PART II—TRIANGLES. PART III—TRANSVERSALS AND PARALLELS. SUMS OF ANGLES OF POLYGONS. PARALLELOGRAMS.

PART I—ANGLES

12. Angles—Definitions and Notation. Suppose that we partially open a fan, or two of the arms of a folding ruler. Suppose also that a line is drawn on each arm of the ruler, starting from the pivot and following the middle of the arm.



Definitions. The figure so formed by two straight lines starting out from the same point is called an **angle.** The point marked A in the figure (the pivot) is called the **vertex** of the angle. The lines AB and AC are called the **arms** of the angle.

Notation. We shall often use a single letter, usually a capital, placed near a point in a figure to designate that point. If we designate a certain point by A, and another point by B, the straight line through these two points is called *the line* AB, or *the line* BA. We would say "the line AB" when the line is drawn from A to B; we would say "the line BA" when the line is drawn from B to A. Often it makes no difference which we use; in other cases a distinction is necessary. See Ch. V, Part II, and Ch. VI, Prob. I.

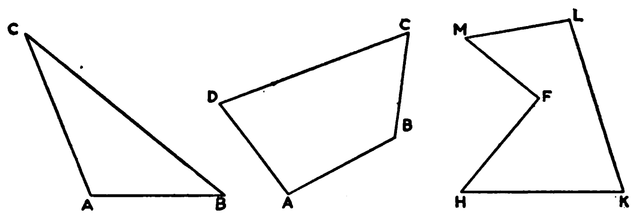
The distinction is of great importance in Algebra, in illustrating the idea of positive and negative numbers.

To designate the angle formed by the lines AB and AC, we say "the angle BAC," or, "the angle CAB." The first means that in opening up the angle we regard AB as a fixed arm and AC as revolving; the second means that AC is the fixed arm and that AB is revolving. Often it makes no difference which notation is used. In either case, to designate an angle, first name a point on the one arm, then name the vertex, and finally name a point on the other arm.

In fixed figures, angles may usually be read either way.

The symbol for the word "angle" is $\angle$; so $\angle BAC$ means "angle BAC."

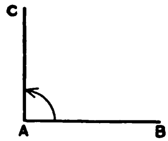
Exercise. In each of the figures below read off the points, lines, and angles marked in it. Draw several figures of your own, letter the points at the ends of each line, and read the angles.



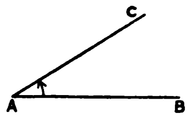
13. Classification of Angles. Angles are classified according to the amount of turning done in separating the arms

In the figure on p. 13 suppose AB to be fixed and AC to be revolving; when AC has revolved half way around, so that it lies just opposite to AB and forms one straight line with it, the angle BAC is called a **straight angle**; half of a straight angle is a **right angle**; if AC turns through less than a right angle it forms with AB an **acute angle**; if AC turns through more than a right angle but less than a straight angle, it forms with AB an **obtuse angle**; more than a straight angle is called a **reflex angle**; a complete turn is called a **perigon.**

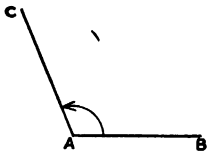
Straight Angle B A C.



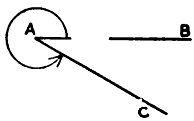
Right Angle B A C



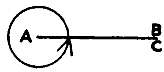
Acute Angle B A C.



Obtuse Angle B A C.



Reflex Angle B A C.



Perigon B A C.

Exercise. Classify each of the angles in the figures on p. 12. Draw a closed figure which shall contain an acute angle, a right angle, an obtuse angle, and a reflex angle.

Definitions.

A **polygon** is a figure bounded by straight lines.

A **convex polygon** is one whose sides, if produced, will not cut the polygon.

The second of the three polygons in article 12 is convex. The third is **concave.**

The angle *HFM* is called a **re-entrant angle.**

A re-entrant angle is always greater than a straight angle.

14. Measurement of Angles. To state the size of an angle we adopt some standard angle as a unit, and say how many of these units are needed to fill the given angle. Two different units are in common use, one called the **degree** the other the **radian.**

Degree Measure of Angles. When a perigon is divided into 360 equal parts, each such part is called a **degree.**

So we have

360 degrees = a perigon.

Then 180 degrees = a straight angle

and 90 degrees = a right angle.

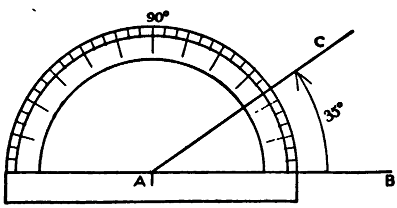
The symbol for degrees is °, so that 10° means "ten degrees."

Exercise. How many degrees in each of the following angles:

(*a*) One fourth of a right angle. (*b*) Two thirds of a straight angle.

(*c*) Two fifths of a perigon. (*d*) Seven twelfths of a perigon.

Angles are usually measured with a protractor (see inside of back cover) as shown in the figure below. To measure



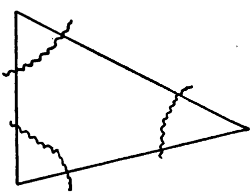
Protractor

a reflex angle, measure its excess over a straight angle, or measure what is lacking to make a perigon.

Exercise 1. Draw ten different angles, some acute, some obtuse, and some reflex. Measure each and write its value on your figure.

Exercise 2. Draw a triangle. Measure each angle. What is their sum? Repeat this with another triangle of different shape.

Exercise 3. Draw a triangle and tear apart as in the figure. Place the three angles with their vertices together, one angle next to the other, without overlapping. What is the sum of the angles of the triangle?



Exercise 4. Repeat Exercise 2, using a quadrilateral, that is, a figure bounded by four straight lines. Do not draw a square or a rectangle, but rather a figure whose sides and angles are quite unequal.

Exercise 5. Repeat Exercise 3, using a quadrilateral.

Exercise 6. Repeat Exercises 2 and 3, using a pentagon, that is, a figure bounded by five straight lines.

Radian Measure of Angles. In this system the unit of measure is a **radian,** instead of a degree as in the system just considered. You can easily make a protractor graduated in radians.

Exercise 1. On stiff paper, or, better, light cardboard, draw a circle with a radius of, say, two inches. Carefully cut it out and mark a point on the circumference or rim. On a good-sized sheet of paper draw a straight line and mark off on it parts, each equal to the radius of the circle. Roll the circle carefully along this line, starting with the marked point on the rim placed at the beginning of the first division on the line. Each time that a point on the rim of the rolling circle reaches a division point on the line mark that point on the rim. Now draw lines from the center of the circle to the points marked on the rim. You then have a series of equal angles, each of which is one radian.

Exercise 2. Define a radian.

Exercise 3. By rolling the circle so that it makes just one complete turn, find approximately how many radians there are in a perigon. You will find a little more than six radians. Estimate the decimal part as well as you can.

15. The Number π. The number of radian units in a perigon is not a whole number, as you found in the last exercise. Nor can this number be expressed either by a fraction or by a terminating decimal. It is a so-called *incommen-*

surable number and by general agreement it is always indicated by 2π, π being a Greek letter called "pi."

We therefore have

$$2\pi \text{ radians} = \text{a perigon} = 360 \text{ degrees.}$$
$$\pi \text{ radians} = \text{a straight angle} = 180 \text{ degrees.}$$

Then $$1 \text{ radian} = \frac{180}{\pi} \text{ degrees.} \quad (\text{About } 57^\circ.3).$$

The number for which π stands, to four decimal places, is 3.1416; less exactly it is $\frac{22}{7}$. It should be remembered that both of these values are only approximate.

16. Definitions. When an angle is generated by the turning of a line **counter clock-wise,** the angle is said to be **positive.**

When an angle is generated by the turning of a line clockwise, the angle is said to be **negative.**

In either case, the first position of the moving line is called the **initial arm** of the angle, and the last position is called the **final arm** of the angle.

17. Postulate V. *All straight angles are equal.*

Corollary. *All right angles are equal.*

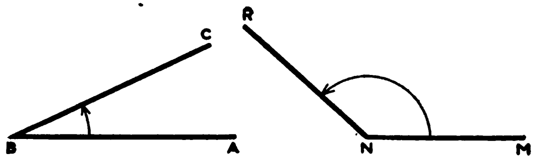
Note. A corollary is a truth that follows immediately from what precedes. The student should always explain the connection.

18. Definition. Two angles whose algebraic sum is equal to a right angle are called **complementary angles.**

Two complementary angles will form a right angle if placed with their vertices together and with the initial arm of the second on the final arm of the first.

When both angles are positive and are placed as above, they will lie adjacent to each other without overlapping; each of them is less than 90°, and hence an acute angle. Of two complementary angles such as 130° and −40°, the second will lie within the first.

Using a protractor draw the complement of each of these angles:

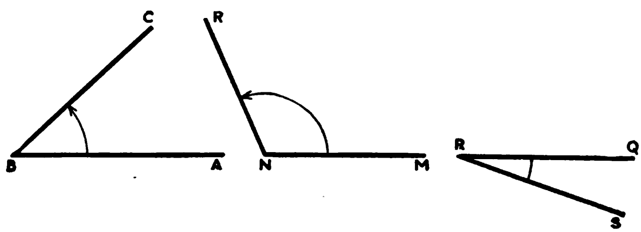


In each figure suppose the final arm of the angle to be turned until it makes a right angle with the initial arm. The angle turned through is the complement of the given angle.

19. Definition. Two angles whose algebraic sum is equal to a straight angle are called **supplementary angles.**

Two supplementary angles will form a straight angle if placed with their vertices together and with the initial arm of the second on the final arm of the first.

Using a protractor draw the supplement of each of these angles:



20. Definition. Two angles whose sum is a perigon are called **conjugate angles.**

If both angles are positive, as 130° and 50°, and are placed to form a straight angle, they will lie adjacent to each other without overlapping. If one of them is negative, as 230° and −50°, the second lies within the first.

Two conjugate angles will form a perigon when placed with their vertices together and with the initial arm of the second on the final arm of the first.

Using a protractor draw the conjugate of each of the above angles.

21. Exercises.

1. What is the complement of an angle of 25°, 15°, 0°, 105°, 115°, 173°, $-36°$, $-128°$, $a°$, $\frac{1}{4}\pi$ radians, $\frac{3}{8}\pi$ radians, $-\frac{1}{8}\pi$ radians.

2. What is the supplement of each of the angles given in Exercise 1?

3. What is the conjugate of each of the angles given in Exercise 1?

4. If an angle is $2a$ degrees, what is its complement?

5. If an angle is $a + 3$ degrees, what is its supplement?

6. If an angle is $5(a - 2)$ degrees, what is its conjugate?

7. If an angle is $\frac{2}{3}\pi + 5$ radians, what is its complement?

8. If an angle is $2a(-3a + 3b)$ degrees, what is its supplement?

9. If an angle is 3 times its complement, what is the size of the angle?

Solve first for the number of units expressed in degrees, second, for the number expressed in radians. (For model for solution, see First Course, pages 114, 115, example 3.)

10. If an angle is $2\frac{2}{3}$ times its complement, what is the size of the angle?

11. If an angle is 200° more than 2 times its complement, what is the size of the angle?

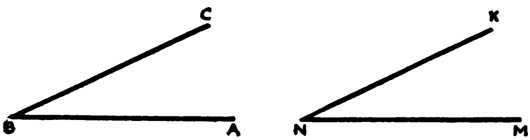
12. If $-15°$ be added to $\frac{3}{4}$ of the supplement of an angle, the sum will be equal to the angle. What is the number of degrees in the angle?

13. What is the size of an angle if its complement is 3 times its supplement?

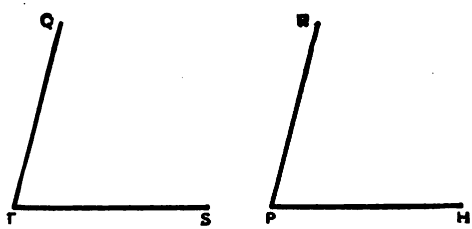
14. Two angles are complementary. If 10° be subtracted from one and added to the other, the two angles will be equal. What is the number of degrees in each angle?

15. Two angles are conjugate. If $\frac{1}{2}\pi$ radians be added to one and the same amount subtracted from the other, the results will be equal. How many radians are there in each angle?

22. Theorem I. *If two angles are equal, their complements are equal.*



We have given the equal angles ABC and MNK; their complements are angles STQ and HPR respectively.



We are to prove that angles STQ and HPR are equal.

Proof. Subtract $\angle ABC$ from a right angle, and subtract $\angle MNK$ from a right angle. The remainders are $\measuredangle STQ$ and HPR respectively. Why?

But these remainders are equal by Ax. 3.

$$\therefore \quad \angle STQ = \angle HPR.$$

The following algebraic proof of this simple theorem should be studied as a step toward more difficult proofs of later theorems.

Let us say that there are a degrees in the $\angle ABC$; then there are a degrees in the $\angle MNK$. Why?

Let c = the number of degrees in $\angle STQ$

and c_1 = the number of degrees in $\angle HPR$.

Then $a + c = 90°$. Why?

Also $a + c_1 = 90°$. Why?

Therefore $a + c = a + c_1$. Why?

Subtracting a from both sides of the equation we have

$c = c_1$. Why?

Since c is the number of degrees in the complement of angle ABC and c_1 is the number of degrees in the complement of angle MNK, we have proved our theorem. State it.

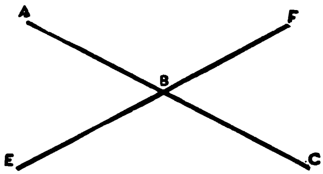
23. Theorem II. *If two angles are equal, their supplements are equal.*

Prove this as in the preceding theorem.

24. Theorem III. *If two angles are equal, their conjugates are equal.*

Prove this as in the preceding theorem.

25. Definition. Vertical Angles.



Angle ABE and angle CBF are called **vertical angles.**

Also angle FBA and angle EBC are called **vertical angles.**

Exercise. Suppose $\angle ABE = 40°$. How many degrees in $\angle FBA$? Why? Then how many degrees in $\angle CBF$? In $\angle EBC$? Give reason for each answer. Answer the same questions when $\angle ABE = n°$. What do you conclude about the four angles at B?

26. Theorem IV. *If two lines intersect, the vertical angles are equal.*

We have given in the figure above the straight lines *AC* and *EF* intersecting at point *B*, forming the vertical angles *FBA* and *EBC*.

We wish to prove that angle FBA equals angle EBC.

Analysis. What is the sum of $\angle FBA$ and $\angle ABE$, no matter what the number of angular units in each of them? Give reason. Also what is the sum of $\angle ABE$ and $\angle EBC$? Give reason. Can you form an equation from these statements? Give reason. Can you subtract the same angle from each member? Why?

Proof. Does this prove the statement to be proved? Write the answers to the above questions in good English.

Prove the other two vertical angles in the above figure equal.

27. Exercises.

1. If an angle is bisected and a line is drawn through the vertex perpendicular to the bisector, it makes equal angles with the arms of the given angle. Prove this statement.

2. Find the value of the angle between the bisectors of two adjacent complementary angles.

3. Find the value of the angle between the bisectors of two adjacent supplementary angles.

4. Find the value of the angle between the bisectors of two adjacent conjugate angles.

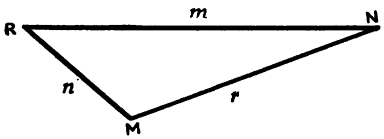
5. How do the bisectors of two vertical angles stand with relation to one another? Prove.

6. Suppose $\angle FBA$ in the figure on p. 20 to be divided into five equal parts by lines drawn from *B*. If these lines are produced back through *B*, show that $\angle EBC$ will be divided into five equal parts. How many degrees in each of these parts if $\angle CBF = 60°$?

7. Draw a figure like that on page 20, making $\angle CBF$ equal to 50°. Suppose a line BX drawn in the angle FBA, so as to make $\angle FBX$ equal to one third of $\angle FBA$; also a line BY in $\angle CBF$, so as to make $\angle CBY$ equal to one-third of $\angle CBF$. Find the number of degrees in $\angle YBX$. Check result by measurement of your drawing. Solve this exercise when $\angle CBF$ is equal to $n°$.

PART II—TRIANGLES

28. Notation. A triangle is named in two different ways, either by placing a capital letter at each vertex and naming these in order, preferably counter-clockwise, or by placing a small letter on each side and naming, preferably counter-clockwise. Thus, we have the triangle RMN or the triangle rmn (figure).

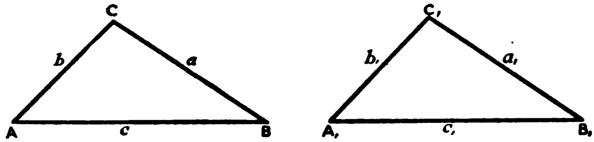


If both vertices and sides are named in the same figure, the letter on the side is usually the small letter corresponding to the capital letter at the opposite vertex. It does not make any difference in naming a triangle at what letter you start.

29. Definition. In geometry figures which, if placed together, can be made to coincide, are called **congruent figures.**

It is not necessary in all cases that the figures actually be placed one on the other in order that we may know that they will coincide and hence are congruent. We shall now try to find other means of knowing.

30. Theorem V. *Two triangles are congruent if they have two sides and the included angle of one equal respectively to the two sides and the included angle of the other.*



Given triangles ABC and $A_1B_1C_1$, such that side a = side a_1, side b = side b_1, and $\angle C = \angle C_1$.

To prove triangle ABC congruent to triangle $A_1B_1C_1$.

Analysis. On your paper draw a triangle ABC, taking $a = 2$ inches, $b = 3$ inches, $\angle C = 40°$. Use a protractor to lay off the angle, and measure off a and b on the arms of this angle.

Draw another triangle $A_1B_1C_1$ with the same values for a_1, b_1, and $\angle C_1$ as for a, b, and $\angle C$.

We wish to compare these triangles as to form and size. To do this cut out the second triangle and lay it on the first with side b_1 on side b seeing that its initial point C_1 is on C, the initial point of b. If the triangles are drawn on thin paper, with rather heavy lines, you can lay one over the other without cutting out.

Then the final point of b_1 is on the final point of b. Why?

What direction will the line a_1 which corresponds to a take? Why?

Where will the final point of a_1 fall? Why?

Where will the remaining line of your triangle c_1 fall? Why? Art. **10**, Post. I.

Then by definition these two triangles are congruent.

The symbolic way for writing this fact is

$$\triangle ABC \cong \triangle A_1B_1C_1.$$

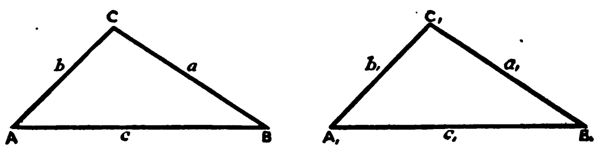
Proof. Imagine the triangles shown in the book to be superposed and answer the questions in the analysis. Therefore the triangles coincide throughout.

$$\therefore \quad \triangle ABC \cong \triangle A_1B_1C_1.$$

State the theorem.

Remark. The sign of congruency, $\cong$, is in reality two signs. The mark above is an *s* written sideways. It stands for similar and means that the figures have the same shape. The equality sign below the *s* means that the figures have the same size. The symbol, then, means that the figures have both the same shape and the same size.

31. Theorem VI. *Two triangles are congruent if they have two angles and the included side of one, equal respectively to the two angles and the included side of the other.*



Given $\triangle ABC$ **and** $\triangle A_1B_1C_1$ **such that side** c = **side** c_1, $\angle A = \angle A_1$, **and** $\angle B = \angle B_1$.

To prove triangle ABC congruent to triangle $A_1B_1C_1$.

Analysis. Draw a triangle ABC, taking $c = 3$ inches, $\angle A = 30°$, $\angle B = 50°$. To do this first draw a line 3 inches long and at its extremities draw $\angle A$ and $\angle B$ using a protractor. Prolong the arms of these angles until they meet.

Draw a second triangle $A_1B_1C_1$ with side c_1, $\angle A_1$, and $\angle B_1$ equal to side c, $\angle A$, and $\angle B$ of the first triangle respectively.

Cut out the second triangle and place it on the first with side c_1 on side c, point A_1 on point A. Then point B_1 falls on point B. Why?

Side a_1 takes the direction of a. Why?

Side b_1 takes the direction of b. Why?

Where must point of intersection C_1 fall. Why?

Proof. Imagine the triangles shown in the book to be superposed and answer the questions in the analysis.

Therefore, since we have shown that the two triangles coincide throughout, we can state that

$$\triangle A_1B_1C_1 \cong \triangle ABC.$$

State the proposition proved.

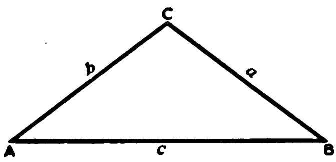
32. Definitions.

A triangle which has two of its sides equal is called an **isosceles triangle.**

A triangle which has three of its sides equal is called an **equilateral triangle.**

A triangle with three unequal sides is called **scalene.**

33. Theorem VII. *The angles opposite the equal sides of an isosceles triangle are equal.*



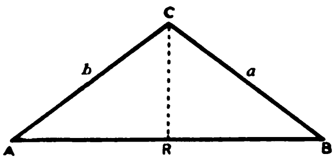
We are given the triangle ABC with the side a equal to side b.

We are to prove that angle BAC equals angle CBA.

Analysis. The only relations which we have yet proved about triangles are the truths of Theorems V and VI. (State them.) So it will be necessary to have two triangles that fulfil one of these relations. These triangles must be such that one will contain one of the angles BAC or CBA, and the other must contain the other of these angles. Why?

To make such triangles draw a line bisecting ∠ ACB. (Use protractor.) Extend this bisector until it intersects side AB.

You thus have two triangles which you can prove congruent. What has this to do with proving ∠ BAC equal to ∠ CBA?



Proof. In ⚠ ARC and RBC,

side b = side a; Why?
side CR = side CR; Why?
∠ ACR = ∠ RCB. Why?

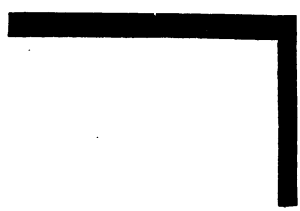
What can you say about ⚠ ARC and RBC? Why?
What can you say about ∠s RAC and CBR? Why?
State the theorem you have proved.

Corollary. *The angles of an equilateral triangle are equal.* Why?

Exercise. Let D be any point on CR in the figure above. Draw AD and BD. Show that △ ABD is isosceles.

34. Definition. One line is said to be **perpendicular** to another when it forms a **right angle** with the other.

Corollary. *At a point in a straight line only one perpendicular can be drawn to the line.*



Exercise. The adjacent figure shows a steel square.

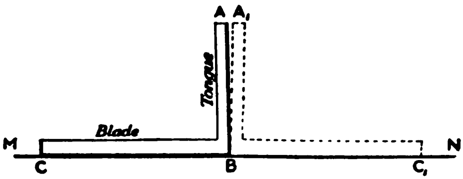
Describe a steel square.

To test a steel square.

Explain how a steel square may

be tested by the method suggested in the following figure. Line AB is drawn along the edge of the tongue with the square in the position ABC, with the blade against the line MN. Line A_1B is drawn with the square in the position A_1BC_1.

What is wrong with this square?



35. Theorem VIII. *Every point in the perpendicular bisector of a straight line is equidistant from the ends of the line.*

Suggestion. Form two triangles by drawing lines from the chosen point on the perpendicular bisector to the ends of the line. Then prove the triangles congruent.

36. Suggestions. The student should observe carefully the steps followed in the proof of a theorem. They are as follows:

(a) Statement of the theorem.

(b) Drawing of a figure, showing the things spoken of in the theorem.

(c) Statement of what is given, called the *hypothesis.*

(d) Statement of what is to be proved, called the *conclusion.*

(e) Analysis, indicating the method of proof.

(f) Proof.

1. Read the statement of the theorem carefully—two or three times if necessary to fix it clearly in your mind.

2. Draw a figure that will contain each thing spoken of in the statement and nothing more.

3. Write down in clear statements what is given or assumed to be true. This is the **Hypothesis.**

4. Write down what you are to prove. This is the **Conclusion.**

5. Analyze, thinking carefully on the following questions:

(a) What am I to prove?

(b) In order to prove this must I prove two things equal, and what are they?

(c) If I must prove two things equal, can I find two triangles of which they are parts? If not, can I make two such triangles by drawing in one or more lines? Can I prove these triangles congruent according to Theorems V or VI?

6. Write down in good English the proof of the theorem.

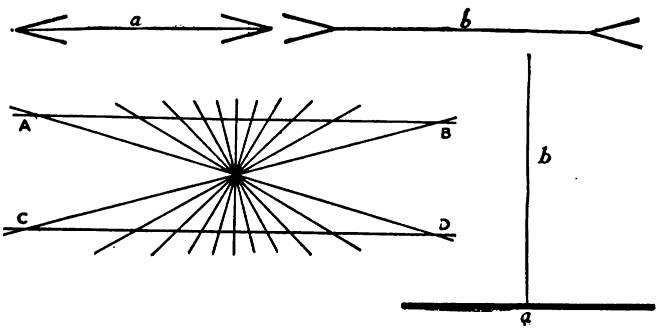
In proving theorems, be careful to draw a general figure, not one that shows merely a special case. For example, in a general theorem about a triangle, do not draw a right triangle or an isosceles triangle.

Be careful not to jump at conclusions from a mere inspection of a figure. Thus, by inspection of the figures below, answer the following questions:

1. Is AB parallel to CD?

2. Is a equal to b?

Check results by measurement.



37. Exercises. Geometric and Algebraic.

1. If perpendiculars are drawn at the ends of a straight line and segments are cut off on these perpendiculars by means of an oblique line passing through the mid-point of the first line drawn, prove that the segments cut on the perpendiculars are equal.

2. If at the mid-point of the side c of the triangle ABC, of which a and b are equal sides, lines are drawn making equal angles with side c and meeting sides a and b, what kind of triangles will be formed? Prove.

3. If the equal angles of an isosceles triangle are bisected and the bisectors are extended until they intersect the opposite sides of the triangle two congruent triangles will be formed. Prove.

4. If a diagonal bisects the opposite angles of a quadrilateral, the diagonal divides the quadrilateral into two congruent triangles. Prove.

5. The line which bisects the angle between the equal sides of an isosceles triangle, bisects the opposite side and is perpendicular to the opposite side. Prove.

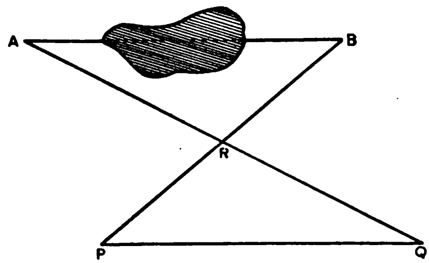
6. Draw any line AB. Draw lines AC and AD, one on one side of AB and the other on the other side, making equal angles with AB. If AC and AD are the same length, prove that BD and BC are the same length.

7. Draw an equilateral triangle ABC. On the three sides mark P, Q, and R, respectively, so that AP equals BQ equals CR. Prove that triangle PQR is equilateral.

8. Draw an isosceles triangle ABC, with A and B as the vertices of the equal angles. From A and B draw perpendiculars to the opposite sides. Prove that these perpendiculars are equal.

9. Draw an equilateral triangle ABC; extend side AB to P, side BC to Q and side CA to R, making BP, CQ, and AR all equal to each other. Prove that triangle PQR is equilateral.

10. If we wish to measure the distance from A to B and there is an obstacle in the way to prevent the direct measurement, show that by taking the following measurements, we can determine AB.



Measure AR, R being any point that can be reached from both A and B.

Measure RQ equal to AR.

Measure BR, and then measure RP equal to BR.

Join PQ and measure it, thus finding the length of AB. Explain.

11. Draw an isosceles triangle ABC with side a equal to side b. Extend side c each way through the vertices to points P and Q respectively, making AP equal to BQ. Prove that $\triangle CPQ$ is isosceles.

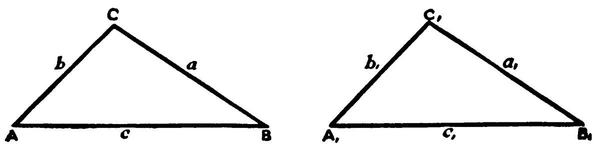
12. An isosceles triangle is drawn. Call it $\triangle PQR$, p and q being the equal sides. Extend side q through the point R a certain length, and mark the point where you stop M. Extend side p through the point R the same distance that you extended q and mark the point of stopping N. Join P to M and Q to N by straight lines. Name two congruent triangles and prove.

13. Of two of the sides of a triangle one is 1 unit less than 2 times the other. If 3 units be subtracted from the one and added to the other the triangle will be isosceles. What is the length of each of the two sides?

14. Two triangles have two sides of one equal respectively to two sides of the other, but the included angle of one 12 degrees more than twice the corresponding angle of the other. If 25 degrees be subtracted from the larger angle and 47 degrees be added to the smaller angle, the triangles will be congruent. What is the included angle of each triangle?

15. In an isosceles triangle, each of the equal angles is twice the third angle. The sum of the three angles is 180°. What are the angles?

38. Theorem IX. *If two triangles have three sides of the one equal respectively to three sides of the other, the triangles are congruent.*

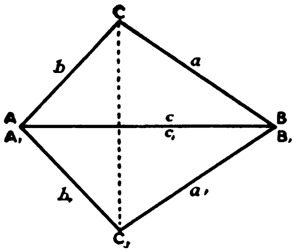


Given two triangles, ABC and $A_1B_1C_1$ having side a_1 = side a, side b_1 = side b, side c_1 = side c.

To prove triangle ABC congruent to triangle $A_1B_1C_1$.

Analysis. If we try to place one of these triangles on the other, in order to see if they coincide, as we did in theorems V and VI, we shall not be able to determine whether the sides coincide or not. Try placing them together with side a_1 upon a, seeing that the initial points fall together. Then the final points will fall together. Why? Now have we any way of knowing whether the side b_1 will take the direction of side b? Do we know the comparative values of the angles between side b and side a, and between side b_1 and side a_1? Therefore can we tell what direction b_1 will take? For this reason we use another plan of placing the triangles together.

Place the triangles with their longest sides together, that is side c_1 upon the side c, but have the two triangles on opposite sides of c.



Draw the line CC_1. We now have two isosceles triangles. Name them. From these we can show an equality of the angles of $\triangle ABC$ and $\triangle A_1B_1C_1$, and so prove the two triangles congruent by means of Theorem V.

Proof.	$\angle C_1CB = \angle BC_1C$.	Why?
	$\angle ACC_1 = \angle CC_1A$.	Why?
therefore	$\angle BC_1A = \angle ACB$.	Why?
	side a = side a_1.	Why?
	side b = side b_1.	Why?

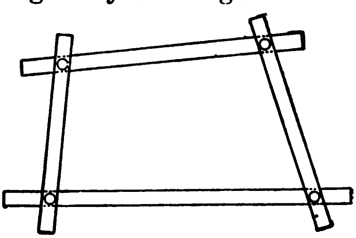
What parts of the two triangles do you now know to be equal? Are the triangles congruent? Why?

State the theorem that you have proved.

39. Exercises. To prove the following exercises proceed as in article **36**, adding Theorem IX to instruction 5, (c).

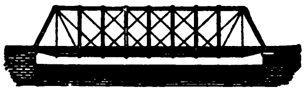
1. State the three cases of congruency of triangles.

2. If two quadrilaterals have four sides of one equal respectively to four sides of the other are they necessarily congruent? Take four strips of paper and fasten as in the figure. Is the figure rigid? Do



the same with three strips. Is the figure rigid? How do you account for the difference? What strip can you add to your quadrilateral to make it rigid?

Now state a theorem for congruency of quadrilaterals and prove.



Panel-truss Bridge.

3. Note the adjacent picture of a bridge truss. What is the purpose of the diagonal beams?

4. Make a five-sided figure and proceed as in Exercise 2.

5. If a line is drawn from the point of intersection of the equal sides of an isosceles triangle to the mid-point of the opposite side, it is perpendicular to the opposite side.

6. In a quadrilateral *ABCD*, the diagonals *AC* and *BD* bisect each other. Show that the opposite sides of the quadrilateral are equal. Apply theorems IV and VI.

7. If the opposite sides of a quadrilateral are equal, either diagonal divides it into two congruent triangles.

8. If two equal isosceles triangles are placed as in the figure on page 32, show that line CC_1 divides the figure into two congruent triangles.

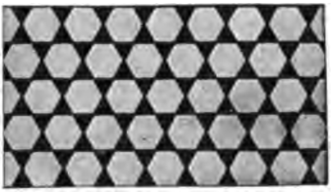
9. Construct an isosceles triangle such that its perimeter shall be 8 inches and its base 3 inches. If the perimeter of an isosceles triangle is n inches, how large can the base be made so that the triangle still may be constructed.

10. Construct a triangle such that its perimeter shall be 9 inches, and such that the second side shall be twice the first side and that the third side shall be equal to the sum of the other two sides.

11. Two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first triangle is 8 units less than twice third side of the second. If the two triangles are to be made congruent without changing the lengths of the equal sides, 6 units must be taken from the third side of the first. Find the lengths of the third side of each at present.

12. Pick out congruent triangles in each of the following designs. State why congruent.

In the first design, each six-sided figure is a regular hexagon, that is, a figure with six equal sides and six equal angles.



Tile Field

Tile Border

40. Constructions. We shall now examine methods for constructing geometric figures without the use of protractor and measuring line.

For this purpose you will need a compass and straight edge, the latter preferably not marked in units.

41. Definitions.

A **circumference** is a closed curved line every point of which is at the same distance from a point within called the **center.**

A part of a circumference is called an **arc.**

The area enclosed by a circumference is called a **circle.**

Any straight line from the center to a point of the circumference is called a **radius.**

42. Assumptions. We shall take for granted the following:

1. Two circumferences will intersect if the distance between their centers is less than the sum of their radii, and greater than the difference of their radii.

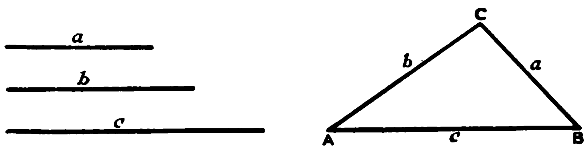
2. The radii of the same circle or of equal circles are equal.

3. If a straight line cuts a circumference once, it cuts it twice.

4. If two circumferences cut each other once, they cut twice.

5. A circumference is determined when its center and radius are known.

43. Problem I. *Given the three sides of a triangle, to construct the triangle.*



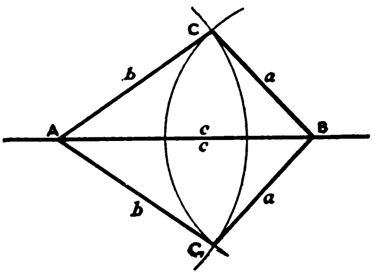
Given the three line-segments *a*, *b*, *c*.

To construct the triangle abc.

Analysis. Suppose the work done and that $\triangle\ ABC$ has the lines *a*, *b*, *c* for its sides. We see that if *AB* is the side of the length *c*, *B* will be one of the vertices and *A* another, while the vertex *C* lies opposite the side *c*. Also the distance from *C* to *A* must equal *b*, and the distance from *B* to *C* must equal *a*. Now every point at distance *b* from point *C* is on the circumference drawn about *C* as center and with a radius *b*. Every point at distance *a* from point *B* is on the circumference drawn about point *B* as a center, with radius *a*. The point which fulfils both of these conditions is on both of the circumferences, so the point of intersection is the vertex *A*.

Construction. Draw a straight line. On this lay off a distance equal to segment *c*. Do this by opening the compass points so that they will just span segment *c*, and then transfer this distance to the line you have drawn. Mark one end *B* and the other *A*. Now set your compass for length *b*. With *A* as a center and *b* as a radius draw a circumference. In like manner with *B* as center and *a* as radius, draw a circumference.

Call the points of intersection C and C_1. Join these points to B and A, and you have two triangles which fulfil the required conditions.



Explain why all other triangles which have these sides are the same size and shape as $\triangle ABC$.

From Assumption 1 of article **42** and the above discussion we have the following theorem:

44. Theorem X. *The sum of two sides of a triangle must be greater than the third side, and the difference must be less than the third side.*

Draw figure and explain.

45. Exercises. Tell whether the following sets of numbers may be the sides of triangles:

1. 2, 4, 6.
2. 5, 10, 3.
3. 256, 174, 89.
4. .06, .1, .078.

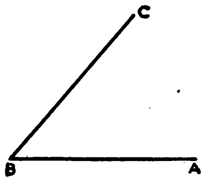
Problem I may also be stated:

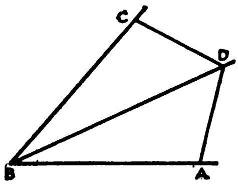
46. Problem II. *To construct two congruent triangles by making their corresponding sides equal.*

Let the student supply *figure, analysis, construction,* and *proof.*

Exercise. The sides of a triangle are 1.5 inches, 2 inches, and 3 inches respectively. Construct a triangle congruent to it. Do the same when the sides of the given triangle are 3, 4, 5 inches respectively.

47. Problem III. *To bisect a given angle.*



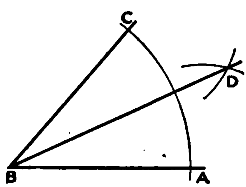


Given angle *ABC.*

To bisect angle ABC.

Analysis. Suppose the work done and that *BD* is the bisector, *D* being any point on it. Suppose that we take the points *A* and *C* so that *BA* equals *BC*, and connect *A* with *D*, and *C* with *D*, forming two triangles. These triangles are congruent. Prove this. Then *AD* and *DC* are equal. Why? Then our problem resolves itself into constructing two triangles with the three sides of the one equal respectively to the three sides of the other. This is done as in Problem II.

Construction. With *B* as a center and with any radius, lay off equal distances *BA* and *BC*.



With *A* as a center and any radius more than one-half the distance *AC*, construct an arc.

With *C* as a center and with the same radius, construct an arc. Call the point of intersection of the two arcs *D*.

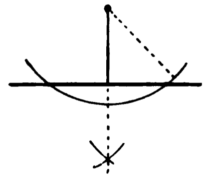
BD is the bisector of $\angle$ *ABC.*

Proof. Give the complete proof.

Corollary 1. *At a given point on a straight line to construct a perpendicular to that line.*

Construction. Bisect the straight angle of which the given point is the vertex.

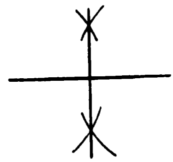
Corollary 2. *From a given point to drop a perpendicular to a given line.*



Construction. Draw two equal line segments from the given point to the given line and bisect the angle between them.

Prove.

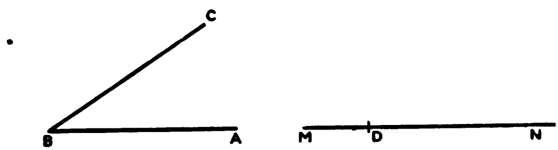
48. Problem IV. *Construct a perpendicular bisector to a straight line-segment.*



Construct as in the adjacent figure.

Prove.

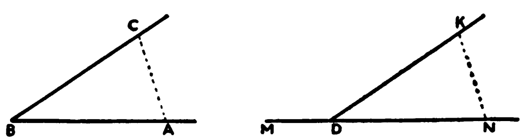
49. Problem V. *At a given point on a given straight line to construct an angle equal to a given angle.*



Given an angle ABC, and the point D on the indefinite line MN.

At point D to construct an angle equal to angle ABC.

Analysis. Suppose the work done and that angle NDK is equal to angle ABC.



Now since points A, C, N, K can have any position on the respective arms of the angles, there is no reason why BC, BA, DN, and DK should not be made the same length. In this event $\triangle$ ACB and NKD would be congruent, and thus NK equal AC. Prove this. So again our problem resolves itself into the construction of two triangles which have the three sides of the one equal to the three sides of the other.

Construction. Make the construction and give the proof.

Problem VI. *To construct a triangle having given two sides and the included angle.*

When a problem is merely stated, the student is expected to supply **figure, analysis, construction** and **proof.**

50. Problem VII. *Construct a triangle, having given two angles and the included side.*

Problems.

1. Construct triangles whose sides are as follows:
   (a) 4 in., 2.5 in., 3 in. (b) 2 in., 5 in., 4 in.
   (c) 10 cm., 14 cm., 10 cm. (d) 15 cm., 6 cm., 10 cm.

2. (a) Construct the triangle whose sides are 3 in., 3.5 in., 4 in., respectively.

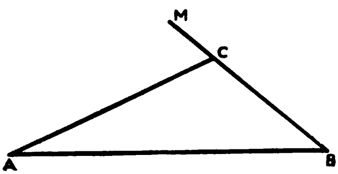
   (b) Erect perpendiculars at the middle points of the sides of this triangle. What do you observe?

   (c) Draw the bisectors of the angles of this triangle. What do you observe?

3. Draw any triangle and repeat (b) and (c) of Exercise 2.

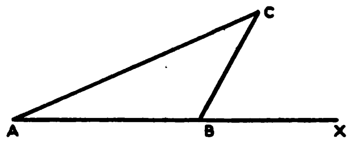
4. Construct a quadrilateral with adjacent sides 3 in. and 4.5 in., respectively, with opposite sides equal, and with the long diagonal equal to 6 inches.

51. Definition. If one of the sides of a triangle is produced, the angle between the side produced and the following side is called an **exterior angle** of the triangle.



Angle MCA is an exterior angle of triangle ABC. Extend other sides and name exterior angles thus formed.

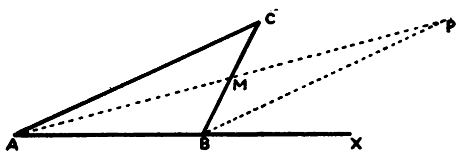
52. Theorem XI. *The exterior angle of a triangle is greater than either of the interior angles not adjacent to it.*



We have given the triangle ABC with the side AB extended through the vertex B, forming the exterior angle XBC.

We are to prove that the exterior angle XBC is greater than either of the interior non-adjacent angles BAC or ACB.

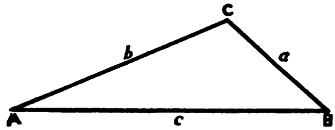
Analysis. To do this we form congruent triangles. We bisect the side BC and call the mid-point M. Draw a line from A to M. Produce AM and on it mark the point P so that MP shall equal AM. Join point B to point P by a straight line.



This gives us $\triangle$ AMC and BPM which we can prove congruent and thus get a comparison of $\angle$ PBC and $\angle$ ACB. By comparing $\angle$ XBC with $\angle$ PBC, we can get its comparison with $\angle$ ACB.

Proof. $\triangle$ AMC and $\triangle$ BPM are congruent. Prove this.
How does $\angle$ PBC compare with $\angle$ ACB? Why?
How does $\angle$ XBC compare with $\angle$ PBC? Why?
Then how does $\angle$ XBC compare with $\angle$ ACB? Why?
By producing side CB through vertex B prove the exterior angle so formed greater than angle A and thus prove the proposition.
State the proposition proved.

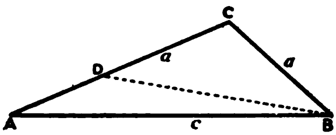
53. Theorem XII. *If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the longer side has the greater angle opposite it.*



Given the triangle ABC, with side b longer than side a.

We are to prove that angle B is greater than angle A.

Analysis. What have we had about angles opposite sides of triangles? (See Theorem VII.) On side b lay off a distance CD equal to side a. This gives us two angles BDC and CBD that we can prove to be equal.



We can now compare $\angle BAC$ with $\angle BDC$, and $\angle CBA$ with $\angle CBD$, and thus get a comparison of $\measuredangle BAC$ and CBA.

Proof. Name the isosceles triangle formed when line BD is drawn. Why is it isosceles?

Name the angles in this triangle which are equal. Why equal?

What is $\angle BDC$ with regard to the $\triangle ABD$?

How does it compare with $\angle A$? Why?

How does $\angle CBA$ compare with $\angle CBD$? Why?

Therefore how does $\angle CBA$ compare with $\angle BAC$? Why?

Assuming that side c is greater than side b, what is true about the angles opposite these sides.

State the proposition you have proved.

Exercises:

1. The interior angles of a triangle are 40°, 60°, and 80°, respectively. Find the value of each of the exterior angles.

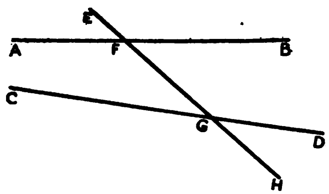
2. If one of the equal angles of an isosceles triangle is 35°, and the sum of the three angles is 180°, find the value of each exterior angle of the triangle.

3. Construct a triangle whose sides shall be 5, 6, and 7 units long, respectively. Which angle is the largest?

4. If two oblique lines are drawn from a given point to a given line, the longer oblique line makes the lesser angle with the given line.

PART III—TRANSVERSALS. PARALLEL LINES. SUMS OF ANGLES OF POLYGONS. PARALLELOGRAMS.

54. Definitions. Draw two straight lines *AB* and *CD*. Cross them with a third straight line *EH*. Let the points of intersection be *F* and *G*.



The angles *GFB* and *FGC* are called **alternate interior angles.**

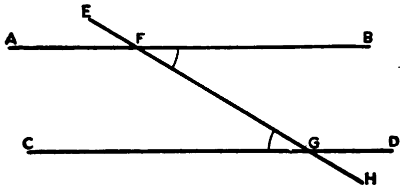
The angles *BFE* and *CGH* are called **alternate exterior angles.**

The angles *BFE* and *DGF* are called **corresponding (or exterior-interior) angles.**

Name other sets of alternate interior angles, alternate exterior angles, and corresponding angles.

55. Theorem XIII. *If two straight lines are cut by a transversal making one set of alternate interior angles equal, then*

(*a*) *the other set of alternate interior angles are equal.*



To construct this figure, first draw two intersecting lines *AB* and *EH* cutting at point *F*. Mark any other point on line *EH*. Call it *G*. At point *G*, using compass and straight edge, make the alternate interior angle to ∠ *GFB* equal to it. (See Problem V, Article **49**.)

Given the straight lines *AB* and *CD* cut by the transversal *EH* at points *F* and *G*, making the angles *GFB* and *FGC* equal.

We are to prove that

(*a*) *the alternate interior* ∠s *DGF and AFG are equal.*

Analysis. In order to prove that ∠ *DGF* equals ∠ *AFG* we shall show that they are the supplements of equal angles, that is we shall show that ∠ *DGF* is the supplement of ∠ *FGC*, and that ∠ *AFG* is the supplement of ∠ *GBF*. This will be sufficient since we made ∠ *FGC* equal to ∠ *GFB*.

Proof. ∠ *DGF* is the supplement of ∠ *FGC*. Why?
∠ *AFG* is the supplement of ∠ *GFB*. Why?
∠ *FGC* is equal to the ∠ *GFB*. Why?
Therefore ∠ *DGF* equals ∠ *AFG*. Why?
State the theorem proved.

In like manner let the student give analysis and proof of the following statements.

(*b*) *the sets of alternate exterior angles are equal;*

(c) *the sets of corresponding angles are equal;*

(d) *the sum of the interior angles on the same side of the transversal is equal to a straight angle;*

(e) *the sum of exterior angles on the same side of the transversal is equal to a straight angle.*

Corollary. *If two straight lines are cut by a transversal, making a set of alternate exterior angles equal, the other sets of angles of the figure are equal.*

There are three more statements of this kind which the student should make and consider.

Exercises:

1. Two lines are crossed by a transversal so that two of the alternate interior angles are 40° and 70° respectively. Find the value of each angle of the figure.

2. In a figure formed by two lines crossed by a transversal, the sum of one pair of alternate interior angles is 150°. What is the sum of the other pair?

56. Definition. If two straight lines in a plane **do not meet** however far they are produced, they are said to be **parallel.**

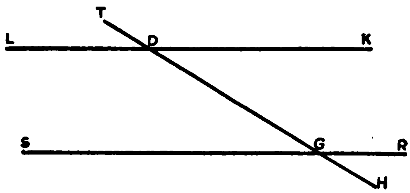
57. Postulate VI. Postulate of Parallels. The following truth is assumed in geometry:

Two intersecting lines cannot both be parallel to the same straight line.

Historical Note. This assumption, or its equivalent as given by Euclid, has been much discussed by mathematicians, especially during the past three centuries. Euclid assumed that, when two lines AB and CD are crossed by a transversal (see figure on page 42), *if the sum of the interior angles on one side of the transversal is less than a straight angle, the lines AB and CD, if produced, will meet on that side of the transversal.*

This was long suspected to be a theorem, capable of proof, and many attempts were made to prove it, but without success. Euclid no doubt made such attempts and his keen insight is shown by the fact that he finally stated it outright as a thing which must be assumed.

58. Theorem XIV. *If two straight lines are cut by a transversal making a set of alternate interior angles equal, the lines are parallel.*



Given the two straight lines LK **and** SR **cut by the transversal** TH **in the points** D **and** G **respectively, making the angle** DGS **equal to the angle** GDK. (Use compass to make the angles equal.)

We are to prove that lines LK and SR are parallel.

Analysis. In order to prove that lines LK and SR are parallel, we must prove that they do not meet no matter how far they are produced. To prove this we shall suppose that they do meet in some distant point, say point P lying to the right of TH, which is too far away to be marked on the page, thus forming $\triangle GPD$ whose exterior angle is DGS. If this were true we should have $\angle DGS$ both equal to $\angle GDK$ and greater than $\angle GDK$ at the same time, which would be impossible. Therefore the lines can not meet, since this would produce an impossible condition.

Proof. Suppose that the lines LK and SR meet at a distant point P to the right.

Name the triangle that will be formed.

What is $\angle DGS$ with reference to this triangle?

What then is the comparison of $\angle$s DGS and GDK? Why?

By hypothesis what is the comparison of $\angle$s DGS and GDK?

What must be the conclusion from these two statements?

Suppose that the lines meet at some distant point to the left, say Q.

By reasoning similar to that just given show that this is impossible.

State the proposition proved.

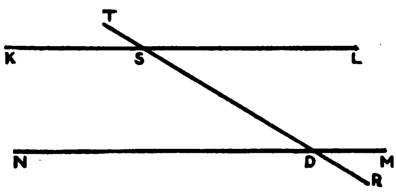
Corollary. *If two straight lines are cut by a transversal, making a set of alternate exterior angles equal, the lines are parallel.*

State other theorems of this kind using the other sets of angles.

Problem VIII. *Through a given point to draw a line parallel to a given line.*

This depends directly on Theorem XIV.

59. Theorem XV. *If two parallel straight lines are cut by a transversal (a) the alternate angles are equal, (b) the corresponding angles are equal, (c) the sum of the interior angles on the same side of the transversal is a straight angle, and (d) the sum of the exterior angles on the same side of the transversal is a straight angle.*



Given the parallel lines KL and NM cut by the transversal TR in the points S and D respectively.

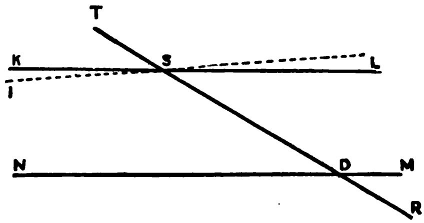
We are to prove that angle MDS equals angle KSD, etc.

Analysis. To prove that $\angle MDS$ is equal to $\angle KSD$, we take a different plan from any which we have given before. Evidently one of the following statements must be true:

either ∠ MDS is less than ∠ KSD,
or ∠ MDS is greater than ∠ KSD,
or ∠ MDS is equal to ∠ KSD.

If we can prove that two of these statements are false, the third must be true.

Taking up the first statement we suppose that ∠ MDS is less than ∠ KSD. If this is true, in ∠ KSD we can lay off an ∠ ISD equal to ∠ MDS, in which case the line IS will lie within ∠ KSD as shown in the figure below. Now since



∠s MDS and ISD are equal, line IS is parallel to line NM. So both the intersecting lines KL and IS are parallel to line NM. Therefore the assumption that ∠ MDS is less than ∠ KSD leads to an impossibility, and we conclude that ∠ MDS is not less than an ∠ KDS.

By a similar line of proof we can show that the second statement is false thus leaving the third one true.

Proof. Suppose $\angle MDS < \angle KSD$, and $\angle ISD = \angle MDS$.

The line IS falls within ∠ KSD. Why?

$\therefore IS \parallel NM$. Why?

But $KL \parallel NM$. Why?

Therefore two intersecting lines KL and IS are parallel to the same line NM, which is impossible. Why? (See postulate of parallels Art. **57**.)

Therefore ∠ MDS cannot be less than ∠ KSD. In like manner prove that ∠ MDS cannot be greater than ∠ KSD.

This proves that $\angle MDS = \angle KSD$.

Prove the other statements by use of Theorem XIII.

60. Theorem XVI. *A line perpendicular to one of two parallel lines is perpendicular to the other.*

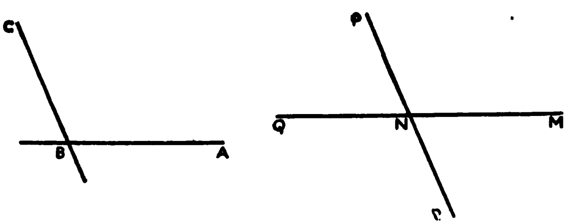
Prove this by making use of Theorem XV and the definition of perpendicular lines.

61. Theorem XVII. *A line parallel to one of two parallel lines is parallel to the other.*

Cross the three lines by a transversal, and apply Theorems XV and XIV.

62. Theorem XVIII. *Two angles whose arms are parallel each to each are either equal or supplementary.*

They are equal when the corresponding arms extend in the same direction from the vertex or in opposite directions from the vertex, and supplementary when one pair of corresponding arms extends in the same direction and the other in the opposite direction.



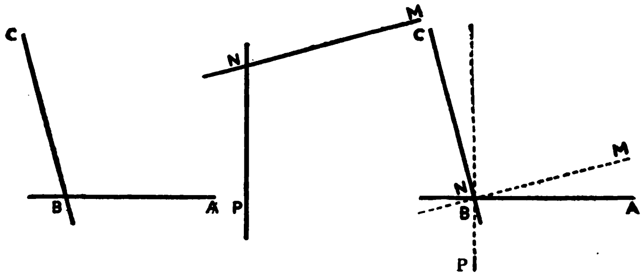
Give this proof by extending the arms until they intersect, and applying Theorem XV and the axiom—*Things equal to the same thing are equal to each other.*

Exercise:

1. Draw an angle ABC, and draw two intersecting lines, one parallel to each arm of the angle. Produce these lines to cut the arms of the angle, forming a quadrilateral. Compare the various angles of the figure, both interior and exterior, with angle ABC.

2. Show how a triangle, cut from cardboard, can be used to draw lines parallel to a given line.

63. Theorem XIX. *Two angles whose arms are perpendicular each to each are either equal or supplementary.*



Give this proof by drawing, through the vertex of one of the angles, lines parallel to the arms of the other as shown in the third figure. Apply Theorems XVIII and I.

64. Exercises depending on Theorems XIII to XIX inclusive.

Read again Article **36**; (c) will now read: If I must prove two angles equal, can I prove that they are

alternate interior angles;
or alternate exterior angles;
or corresponding angles of parallel lines?

If I must prove lines parallel, can I prove that, being cut by a transversal,

the alternate interior angles
or the alternate exterior angles
or the corresponding angles

are equal? If I must prove these equal, can I prove that they are the corresponding angles of congruent triangles? Or can I prove them equal to the same angle?

1. In figure under Theorem XV, what is the number of degrees in each of the angles if $\angle DSL$ is 20 degrees? 108 degrees? a degrees? $(a + b)$ degrees? $a(a + 3b)$ degrees?

2. In the same figure how many radians in each of the angles of the figure if ∠ RDM is $\frac{1}{2}\pi$ radians? $\frac{2}{3}\pi$ radians? $a + \frac{1}{5}\pi$ radians? $2(r + \frac{1}{c})\pi$ radians? $(\frac{a}{c} - \frac{1}{a})\pi$ radians? $1\frac{1}{2}$ radians? $a + \frac{r}{t}$ radians? Express each answer as an improper fraction when possible.

3. Find the number of degrees in each angle of the same figure, if ∠ TSL is $2\frac{1}{2}$ degrees more than 4 times ∠ NDS.

4. If ∠ SDN is $\frac{1}{5}$ ∠ KSD, find the number of radians in each angle of the same figure.

5. If ∠ SDN is $\frac{1}{r}$ part of ∠ KSD, find the number of radians of each angle of the figure.

6. If ∠ LST is $\frac{1}{s}$ degrees more than r times ∠ RDM, how many degrees are there in each of the angles of the figure?

7. Write three problems involving Theorems XIII, XV.

8. Two lines AB and CD are cut by a transversal DE in the points F and G respectively, forming the alternate interior angles GFB and FGC. ∠ GFB is 8 degrees more than 2 times ∠ FGC. If ∠ GFB were made 28 degrees less and ∠ FGC were made 40 degrees more the lines would be parallel. Find the number of degrees in each angle as they are drawn.

9. Two lines are crossed by a transversal making one of the two alternate exterior angles equal to the $\frac{m}{n}$ part of the other. If $\frac{m}{n}$ part of a right angle is added to the one and $\frac{n}{m}$ part of a right angle is subtracted from the other, the lines will be parallel. Find the number of degrees in each angle.

10. If PQR is an isosceles triangle with side p equal to side q, the bisector of the exterior angle formed at R by producing side q is parallel to side PQ.

11. If a point is placed between two parallel lines, and through the point two intersecting lines are drawn cutting the parallel lines, two mutually equiangular triangles are formed.

12. If two parallel lines are cut by a transversal, and each angle of a pair of alternate interior angles is bisected, the bisectors are parallel.

13. If two parallel lines are cut by a transversal, and the interior angles on the same side of the transversal are bisected, the bisectors are perpendicular to each other.

14. Cut a triangle from stiff paper. By sliding it along a straight line show how it may be used to draw parallel lines.

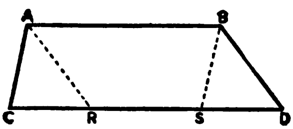
(A right-angled triangle is commonly used for this purpose.)

15. If through any point on the hypotenuse of a right-angled triangle a line is drawn parallel to one of the sides, it is perpendicular to the other.

16. If through any point on the hypotenuse of a right-angled triangle, a line is drawn perpendicular to one of the sides it is parallel to the other.

17. *M* is any point on the hypotenuse of a right triangle *ABC* of which *CA* is the hypotenuse. Through *M* two lines are drawn, one parallel to side *AB* cutting *BC* in point *R*, and one perpendicular to *CA* cutting *BC* (or *BC* produced) in *S*. Prove that ⚠ *ABC* and *MSR* are mutually equiangular.

18. In the figure $AB \parallel CD$, $AR \parallel BD$, and $BS \parallel AC$. Prove that ⚠ *CRA* and *SDB* are mutually equiangular.



19. Construct a triangle, equiangular with a given triangle, by drawing lines parallel to the sides of the given triangle. Prove.

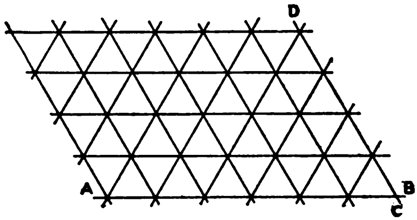
20. Bisect the four angles of a quadrilateral whose opposite sides are parallel, and produce the bisectors to intersect, forming a new quadrilateral. Show that all its angles are right angles.

21. In a quadrilateral with opposite sides parallel, a diagonal bisects a pair of opposite angles. Prove the four sides equal.

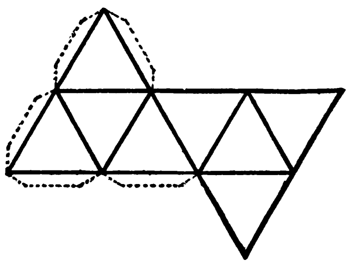
22. Construct an angle of 60°.

Suggestion. Construct an equilateral triangle, and assume that the sum of the three angles is 180°.

23. Construct two lines intersecting at an angle of 60°. Call one AB and the other CD. Regarding AB as a transversal lay off a number of equal segments on it. Through the points of division construct lines parallel to CD. Regarding CD as a transversal lay off a number of segments equal to those on AB. Through the points of division construct lines parallel to AB. By drawing lines through the points of intersection you will have a design of equilateral triangles.

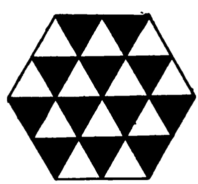


24. Construct this design, formed of equilateral triangles. It may be used as pattern for making a model of one of the so-called *regular solids*, namely, the *regular octahedron*.



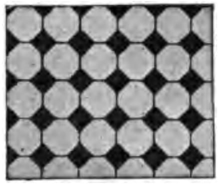
25. By constructing parallel lines and darkening portions bring out the adjacent design.

26. As in Exercise 25, for this tiling pattern.



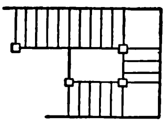
27. Make a design of your own, based on equilateral triangles.

28. Outline the following designs.

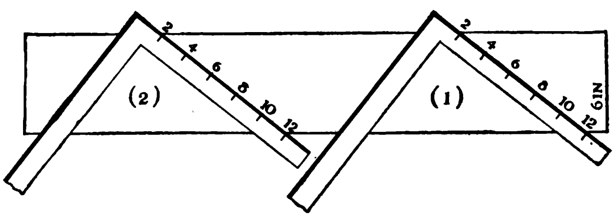


29. By constructing parallel lines and darkening portions bring out the adjacent design.

30. Make drawing showing plan of a stairway having four steps to the first landing, then four steps to the second landing, then six steps to the top.

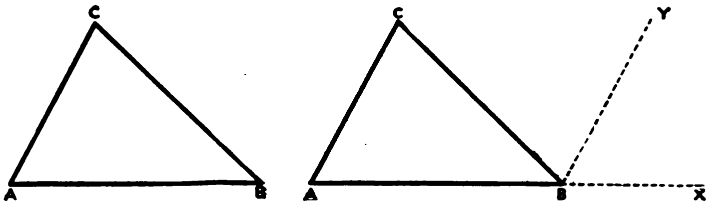


31. To divide a board lengthwise into a given number of equal parts; for example, to divide a 6-inch board into 5 equal parts.



Place the square or ruler across the board as shown in the figure, so that the two-inch and the twelve-inch mark shall fall on the edges of the board. Mark on the board at the readings 4, 6, 8, 10. Move the square to a second position, as in (2). Again mark the points 4, 6, 8, 10. Join the marks with corresponding numbers. Explain why.

65. Theorem XX. *The sum of the angles of a triangle is equal to a straight angle.*



Given the triangle ABC.

To prove that $\angle A + \angle B + \angle C =$ *a straight angle.*

Analysis. To do this, we shall construct three angles which shall equal respectively the angles of the triangle, and whose sum shall equal a straight angle. To do this produce the side AB through B to X, as in the figure. Through B draw a line BY parallel to side AC. This gives three angles whose sum is a straight angle. If we can prove $\measuredangle XBY$, YBC, and CBA equal respectively to $\measuredangle BAC$, ACB, and CBA, we shall have proved our theorem.

Proof. $\angle XBY = \angle BAC$. Why?

$\angle YBC = \angle ACB$. Why?

$\angle CBA = \angle CBA$. Why?

$\angle XBY + \angle YBC + \angle CBA =$ a straight angle.

$\therefore$ $\angle BAC + \angle ACB + \angle CBA =$ a straight angle. Why?

State the theorem proved.

Corollary 1. *The exterior angle of a triangle is equal to the sum of the interior angles not adjacent to it.*

Corollary 2. *If a triangle has one right angle or one obtuse angle, each of the other angles is acute.*

Corollary 3. *If two angles of a triangle are equal respectively to two angles of another triangle, the third angles are equal.*

Corollary 4. *If two triangles have two angles and any side of one equal respectively to two angles and the corresponding side of the other, the triangles are congruent.*

Corollary 5. *If two right triangles have an acute angle and a side of one equal respectively to an acute angle and the corresponding side of the other, the triangles are congruent.*

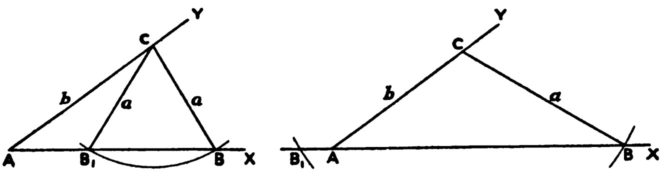
66. Theorem XXI. *If a triangle has two of its angles equal, the sides opposite are equal and the triangle is isosceles.*

To prove this divide the triangle into two triangles by drawing a line from the vertex of the third angle perpendicular to the side included between the equal angles.

67. Problem IX. *Given two angles of a triangle, to construct the third angle.*

68. Problem X. *Given two angles of a triangle and the side opposite one of them, to construct the triangle.*

69. Problem XI. *Given two sides of a triangle and the angle opposite one of them, to construct the triangle.*



Given the angle A and the sides a and b of a triangle, the side a being opposite angle A.

To construct the triangle.

Construction. Draw XAY equal to the given angle. On AY lay off AC equal to b.

With C as a center and a radius a draw arc BB_1.

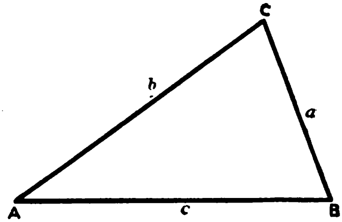
Then the first figure shows two possible triangles, ABC and AB_1C.

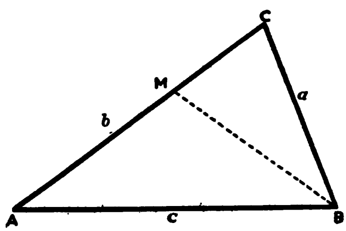
The second figure, where a is longer than b, shows only one possible triangle. Give proof in each case.

Let the student draw figures with shorter values of a. How long must a be to make the construction possible?

Study also the cases where angle A is an obtuse angle.

70. Theorem XXII. *If two angles of a triangle are unequal, the sides opposite are unequal and the greater angle has the longer side opposite it.*





Given the triangle ABC with the angle B greater than angle A.

To prove that side b is longer than side a.

Analysis. Since we now know that if two angles of a triangle are equal the sides opposite are equal, we draw in the line BM making $\angle\ MBA$ equal to $\angle\ BAC$. Line BM will fall between sides c and a, because $\angle\ CBA$ is greater than $\angle\ A$. Now line BM equals AM; also $CM + MB$ is greater than CB, since a straight line is the shortest distance between two points. So we have proved the theorem when we have proved the statements made in our analysis.

Proof. $BM = MA$. Why?

$CM + MB > a$. Why?

$\therefore\ CM + MA > a$. Why?

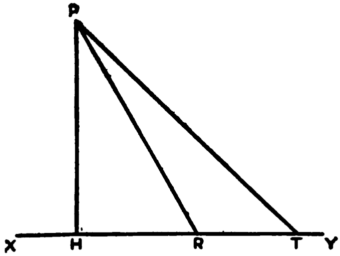
$\therefore\ b > a$. Why?

State the proposition proved.

Corollary 1. *The hypotenuse of a right-angled triangle is the longest side of the triangle.*

Corollary 2. *Of all the lines drawn from an external point to a given straight line the perpendicular is the shortest.*

Corollary 3. *Of two oblique lines drawn from an external point to a given straight line, the one which cuts the greater distance from the foot of the perpendicular is the longer.*



Suggestion. In the figure let PH be the perpendicular from P on line XY. $\angle TRP$ is obtuse because it is an exterior angle to $\triangle HRP$. Hence PT is longer than PR, since it lies opposite the greatest angle in $\triangle RTP$.

71. In the following exercises observe the suggestions below:

If, the relations of angles being given, it is required to find the values of the angles, form an equation by applying one of the following:

Theorem. *The sum of the angles of a triangle equals a straight angle.*

Or apply **Corollary.** *The sum of the acute angles of a right-angled triangle equals a right angle.*

Or apply **Axiom.** *Things equal to the same thing are equal to each other.*

For examples and illustrative solutions, see First Course, pages 37 and 38, Exercises 9 to 21.

To prove two lines equal, prove that they are corresponding sides of congruent triangles; or prove that they are opposite the equal angles of an isosceles triangle.

To prove that a triangle is isosceles, prove that two of its angles are equal, or that two of its sides are equal.

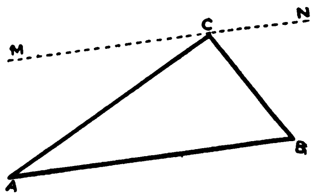
To prove that two angles are equal, prove
that they are the alternate interior angles;
or they are the alternate exterior angles;
or they are the corresponding angles of parallel lines;
or they have their arms parallel each to each;
or they have their arms perpendicular each to each;
or they are vertical angles;
or they are complements of equal angles;
or they are equal to equal angles;
or they are halves of equal angles;
or they are corresponding angles of congruent triangles.

To prove two triangles congruent, prove that
they have two sides and the included angle of the one equal respectively to two sides and the included angle of the other.
or they have two angles and the included side of the one equal to two angles and the included side of the other.
or they have three sides of the one equal respectively to the three sides of the other.
or they have two angles and an opposite side of the one equal to two angles and an opposite side of the other.

To prove two triangles mutually equiangular, prove that they have two angles of one equal respectively to two angles of the other; then the third angles are equal.

72. Exercises.

1. Prove that the sum of the angles of a triangle is a straight angle by using the adjacent figure, in which the line MN is drawn parallel to the side AB.



2. The sum of the acute angles of a right triangle is $\frac{1}{2}\pi$ radians.

3. If one angle of a right triangle is 45°, the triangle is isosceles.

4. If one acute angle of a right triangle is twice the other, find the number of degrees in each angle of the triangle. Show that such a triangle may be formed by bisecting an equilateral triangle.

5. In the triangle of Exercise 4 prove that the hypotenuse is twice the side opposite smaller angle.

Suggestion. Draw a line through the vertex of the right angle cutting the hypotenuse in such a way that two isosceles triangles will be formed. Or, bisect an angle of an equilateral triangle.

6. If the largest of the three angles of a triangle is 31 degrees more than 3 times the smallest, and 33 degrees more than 2 times the intermediate angle, what is the number of degrees in each angle of the triangle?

7. If the first angle of a triangle is 30° more than twice the second, and the third is 12° more than the sum of the first and second, what is the number of degrees in each angle of the triangle, both interior and exterior?

8. One of two angles of a triangle is $15\frac{4}{5}$ degrees less than $\frac{2}{5}$ of the other. If the side included by these angles be turned so that the smaller angle becomes 37 degrees more and the larger angle is 37 degrees less, the triangle will be isosceles. Find the number of degrees in each angle of the triangle.

9. In a right triangle one of the acute angles is 6° less than $\frac{1}{3}$ of the other. How many degrees must be taken from the larger and added to the smaller angle to make the hypotenuse twice the shortest side of the triangle.

10. One of the angles of an isosceles triangle is $\frac{a}{b}\pi$ radians; what is the number of radians in each angle of the triangle both interior and exterior.

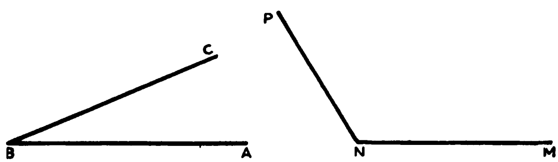
11. If one of the angles of a triangle is r degrees more than a times another, and the difference between the two is $\frac{q}{p}$ part of the third, find the number of degrees in each angle of the triangle, both interior and exterior.

12. The angle between the equal sides of an isosceles triangle is m degrees. A line is drawn to divide it into two parts such that p times the greater after it is increased by p degrees is equal to r times the less after it is diminished by $\frac{r}{2}$ degrees. Find the number of degrees in each angle of the figure.

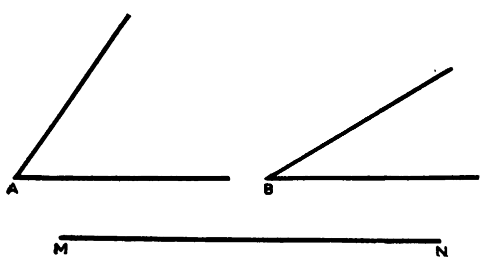
13. Write a problem concerning the angles of a triangle using special relations.

14. Write a problem concerning the angles of a triangle using general numbers.

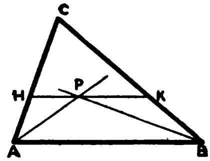
15. If the angles here shown are two angles of a triangle, construct the third angle.



16. If $\measuredangle$ A and B are the angles of a triangle, and line MN is the side opposite $\angle A$ construct the triangle.



17. If through a point on the bisector of an angle a line is drawn parallel to one arm of the angle and cutting the other, an isosceles triangle is formed.



18. If ∠s A and B of △ ABC are bisected by lines intersecting in P, and a line HK is drawn through P parallel to AB, the line HK is equal to the sum of the segments AH and BK.

19. If from any point in the bisector of an angle lines are drawn perpendicular to the arms of the angle two congruent triangles are formed, and hence the perpendiculars are equal. This proposition may be worded:

Any point on the bisector of an angle is equidistant from the arms of the angle.

20. If the equal angles of an isosceles triangle are bisected, and the bisectors are extended to intersect the opposite sides, two sets of congruent triangles are formed.

21. The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.

22. If, in an isosceles triangle, the two equal angles are bisected, and the bisectors are extended to intersect, a new isosceles triangle is formed.

23. In Exercise 22 the angle at the vertex of the first triangle is 100°; what is the angle at the vertex of the new triangle?

24. In Exercise 22, suppose that the angle at the vertex of the original triangle is equal to one-half of one of the equal angles. Find the angles of the original triangle and of the new triangle.

25. In Exercise 20, suppose each of the equal angles to be 70°; find all the angles formed when the bisectors are drawn.

26. In the isosceles triangle PQR, side p being equal to side q, the side q is produced through the vertex R to the point K so that RK equals q. Line QK is drawn. Prove that △ RQK is isosceles. Prove that △ PQK is a right triangle.

27. If, in a right triangle, a line is drawn from the vertex of the right angle perpendicular to the hypotenuse, two triangles are formed which are mutually equiangular to the given triangle and to each other.

28. If $\triangle MNO$ is isosceles with $MO = NO$, and a line is drawn perpendicular to MN cutting MN in R, MO in Q and NO produced in P, prove that $\triangle QOP$ is isosceles.

73. Definitions.

A polygon of n **sides** is in general called an n-**gon.**

Special names are:

3 *sides:*	*triangle.*
4 *sides:*	*quadrilateral.*
5 *sides:*	*pentagon.*
6 *sides:*	*hexagon.*
7 *sides:*	*heptagon.*
8 *sides:*	*octagon.*
9 *sides:*	*nonagon.*
10 *sides:*	*decagon.*
12 *sides:*	*dodecagon.*

A polygon is **regular** when its sides are equal and its angles are equal.

A **diagonal** of a polygon is a line drawn from one vertex to any other vertex not adjacent to it.

74. Theorem XXIII. *The sum of the angles of an n-gon is equal to* $(n - 2)$ *straight angles.*

In order to prove this we will examine several different polygons. Draw a quadrilateral. Draw one of its diagonals. Into how many triangles does the diagonal divide the quadrilateral? What is the sum of the angles of each triangle? What is the sum of the angles of the two triangles and hence of the quadrilateral?

Draw a pentagon. Draw the diagonals from one vertex. Into how many triangles do the diagonals divide the pentagon? What is the sum of the angles of each triangle? What is the sum of the angles of the three triangles and hence of the pentagon?

Repeat this experiment with a hexagon; with a heptagon; with a nonagon.

Make a table of your answers to the above, thus:

Number of sides	Number of triangles	Sum of angles in one triangle	Sum of angles of the figure
4	2 = 4 − 2	1 st. angle	(4 − 2) st. angles
5	3 = 5 − 2	1 st. angle	(5 − 2) st. angles

Do you find that you can tell the number of triangles into which a 20-angled figure can be divided by drawing the diagonals from one vertex? A 38-angled figure?

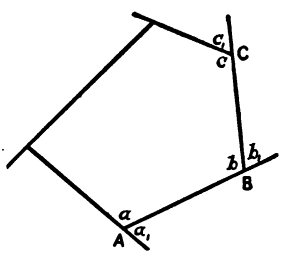
What is the sum of the angles of the 20-angled figure? Of the 38-angled figure?

Into how many triangles will the diagonals of an n-angled figure divide the figure? What then is the sum of the angles of an n-angled figure?

Give a complete proof of the theorem.

State the theorem that you have proved.

75. Theorem XXIV. *The sum of the exterior angles of a convex polygon is equal to a perigon.*



Given the polygon $ABC \dots$ **having** n **sides.**

Let a = the number of angular units in the interior angle at A,

a_1 = the number of angular units in the exterior angle at A,

b = the number of angular units in the interior angle at B,

b_1 = the number of angular units in the exterior angle at B, and so on.

To prove that $a_1 + b_1 + c_1 +$ etc. = a perigon.

Analysis. To prove this we find the sum of the interior and exterior angles at each vertex, and then by multiplying by the number of vertices we shall have the sum of the interior and exterior angles of the polygon. Subtracting the sum of the interior angles we arrive at the sum of the exterior angles.

Proof. What is the value of $a + a_1$?
What is the value of $b + b_1$?
What is the value of $c + c_1$?

Since n is the number of vertices of the polygon, the sum of the interior and the exterior angles is

n straight angles. Why?

The sum of the interior angles is

$(n - 2)$ straight angles. Why?

Then the sum of the exterior angles is the amount that must be added to $(n - 2)$ straight angles to get n straight angles, or it is

$n - (n - 2)$ straight angles,

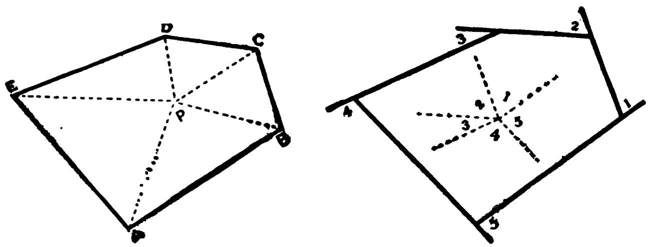
which equals 2 straight angles, or a perigon.

State the proposition proved.

76. Exercises. Angles of Polygons.

1. What is the number of sides of a polygon if the sum of its interior angles is equal to the sum of its exterior angles.

2. Prove Theorem XXIII by placing a point within the polygon, and joining it to the vertices of the polygon as shown in the first figure.



Again prove Theorem XXIII by means of the second figure.

3. What is the number of degrees in each angle of an equiangular hexagon?

4. How many sides has a polygon the sum of whose interior angles is π radians?

5. In a pentagon the second angle is 5 degrees larger than the first; the third is twice the size of the second; the fourth is twice the size of the third; the fifth is two-thirds the size of the fourth. What is the number of degrees in each angle of the pentagon?

6. In a pentagon the first two angles are equal; the third is 14 degrees less than 3 times the first; the fourth is 4 times the difference between the third and second; the fifth is 39 degrees. What is the number of degrees in each angle of the pentagon?

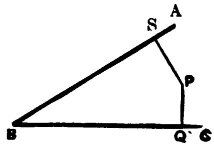
7. In a hexagon the second angle is 24° more than the first; the third is 24° less than twice the second; the fourth is twice the first; the fifth is one-half the first; the sixth is 276°. What is the number of degrees in each angle of the hexagon, both interior and exterior?

8. Write a problem by substituting general numbers instead of the specific numbers in Exercise 6 and solve. Check

your answers by substituting the special values found in Exercise 6, and comparing results with answers to Exercise 6.

9. Repeat Exercise 8 using Exercise 7 as the basis of problem.

10. The point P lies in $\angle CBA$. PQ and PS are drawn perpendicular to the arms respectively. Prove that $\angle SPQ$ is the supplement of $\angle CBA$. Compare with Theorem XIX.



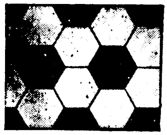
11. What is the value of each angle of a regular triangle? What is another name for a regular triangle? What is the value of each angle of a regular quadrilateral? What is another name for a regular quadrilateral? What is the value of each angle of a regular pentagon? Of a regular hexagon? Of a regular heptagon? Of a regular n-gon?

12. What is the value of each exterior angle of the figures mentioned in Exercise 11?

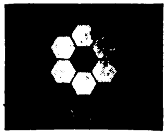
13. What is the ratio of the interior and exterior angle of each figure mentioned above?

Using this ratio for the regular n-gon as a formula, find the ratio of the interior and exterior angle of a regular polygon of 12 sides. Of 20 sides. Of 50 sides. Does the ratio grow greater or less as we increase the number of sides?

14. Draw the bisectors of the interior angles of a regular hexagon. Show that they meet in a point equidistant from the vertices, and divide the hexagon into six equilateral triangles.



15. Which of the regular figures can be arranged about a point so as to form designs for wall-paper or linoleum? What combinations can be so used? Make some designs using regular figures. (In the adjacent design three regular hexagons come together at each vertex. The design of Ex. 14 contains regular hexagons and triangles.)



77. Definitions.

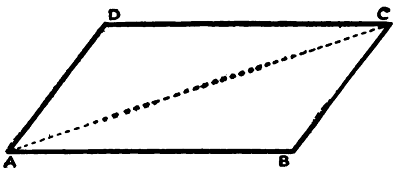
A **parallelogram** is a quadrilateral whose **opposite sides are parallel.**

A **rectangle** is a parallelogram which has one **angle a right angle.**

A **square** is a rectangle with its **adjacent sides equal.**

A **rhombus** is a parallelogram whose **sides are all equal,** but which contains **no right angle.**

78. Theorem XXV. *If a convex quadrilateral has its opposite sides equal, the figure is a parallelogram.*



Given the quadrilateral $ABCD$ **with**

$$\text{side } AB = \text{side } CD,$$
$$\text{side } DA = \text{side } BC.$$

To prove that quadrilateral ABCD is a parallelogram.

Analysis. In order to prove that quadrilateral $ABCD$ is a parallelogram we must prove that AB is parallel to CD, and that BC is parallel to DA. To prove that these lines are parallel we cross them by a transversal AC. Now we can prove $AB \parallel CD$ if we can prove that the alternate interior angles BAC and DCA are equal, and we can prove that $BC \parallel AD$ if we can prove that the alternate angles ACB and CAD are equal. We can prove these angles equal if we can prove ≜ ABC and ACD congruent. We can prove these triangles congruent if we can prove that three sides of the one are equal respectively to three sides of the other.

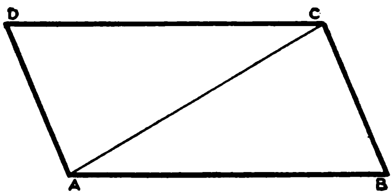
Proof. Let the student give this proof complete.

79. Theorem XXVI. *If a convex quadrilateral has two of its sides equal and parallel, the figure is a parallelogram.*

Give drawing, analysis, and proof as in the above Theorem.

Suggestion. In the quadrilateral $ABCD$ let AB and CD be parallel and equal. Will it then be sufficient to prove $BC = DA$? Can you do this by congruent triangles?

80. Theorem XXVII. *A diagonal divides a parallelogram into two congruent triangles.*



Given the parallelogram $ABCD$, **with the diagonal** AC, **forming triangles** ABC **and** ACD.

To prove triangles ABC and ACD congruent.

Analysis. We can prove that $\triangle ABC \cong \triangle ACD$ if we can prove that $\angle BAC = \angle ACD$, and that $\angle ACB = \angle CAD$, since line AC is common to both. We can show that $\angle BAC = \angle ACD$ if we can show that they are alternate interior angles of the parallel lines AB and CD. We can show that $\measuredangle ACB$ and CAD are equal if we can show that they are the alternate interior angles of the parallel lines BC and DA.

Let the student make this proof complete.

Corollary 1. *The opposite sides of a parallelogram are equal.*

Corollary 2. *The opposite angles of a parallelogram are equal.*

Corollary 3. *The sum of two consecutive angles of a parallelogram is equal to a straight angle.*

Corollary 4. *Each angle of a rectangle is a right angle.*

81. Theorem XXVIII. *The diagonals of a parallelogram bisect each other.*

Let the student give figure, analysis and proof.

82. Exercises.

1. If two lines bisect each other, the figure formed by joining their ends is a parallelogram.

2. The diagonals of a rectangle are equal.

3. The diagonals of a square are perpendicular to each other.

4. The diagonals of a rhombus are perpendicular to each other.

5. If the four interior angles formed by two parallel lines cut by a transversal are bisected, the four bisectors, extended to intersect, form a parallelogram.

6. Prove that the parallelogram of Exercise 5 is a rectangle.

7. The bisectors of the angles of a parallelogram extended to intersect form a rectangle.

8. On the diagonal AC of the parallelogram $ABCD$ equal distances AP and CQ are laid off from the vertices A and C. The lines BQ and DP are drawn. Prove that the figure $PBQD$ is a parallelogram.

9. One of the angles of a parallelogram is $\frac{a}{b}\pi$ radians. What is the size of each of the others? If one is $(\frac{m}{r} - \frac{r}{q})\pi$ radians, what is the size of each of the others?

10. How many radians would have to be added to the angle given in the first question of Exercise 9, in order that the parallelogram be a rectangle? How many in the second question?

11. One of the angles of a parallelogram is m degrees more than $\frac{r}{s}$ part of one of the others. Find the number of degrees in each angle of the parallelogram. How many degrees if $m = 30°, r = 3, s = 4$.

12. The longer side of a rectangle is k cm. less than $\frac{h}{k}$ part of the shorter. If h cm. be subtracted from the longer and added to the shorter, the figure will be a square. Find the length of each side of the rectangle.

13. Make a square by hinging together four strips of paper. Swing it over until one angle is $\frac{3}{4}$ the size of the angle of the square. What is the size of each angle of the new figure? Give answer in radians and also in degrees. What is the new figure called? What truths hold for the new figure that did not hold for the square. State truths that hold for both figures.

14. If a line is drawn through the point of intersection of the diagonals of a parallelogram, parallel to one side, it will bisect two sides of the parallelogram.

Definitions.

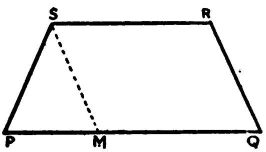
A **trapezoid** is a quadrilateral which has **two parallel sides,** the other two sides being not parallel.

The **parallel sides** are frequently spoken of as the **bases** of the trapezoid.

If the non-parallel sides are equal, the trapezoid is said to be **isosceles.**

15. The angles at the same base of an isosceles trapezoid are equal.

Suggestion. If $PQRS$ be the trapezoid, draw $SM \parallel RQ$, and show that $\measuredangle QPS$ and PQR are both equal to $\angle SMP$. This proves that the angles at the lower base are equal. Now show that the angles at the upper base are equal.

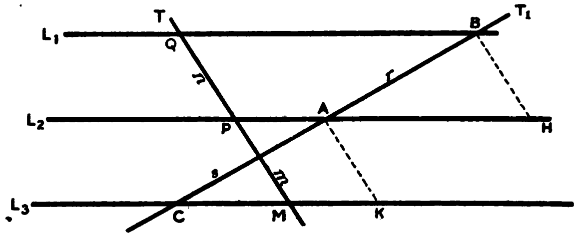


16. The diagonals of an isosceles trapezoid are equal.

17. Construct an isosceles trapezoid whose bases are 3 inches and 4 inches respectively and whose perimeter is 17 inches.

MORE THAN TWO PARALLEL LINES CUT BY TRANSVERSAL

83. Theorem XXIX. *If a system of parallels cuts equal parts on any transversal, it cuts equal parts on every transversal.*



Given the parallel lines L_1, L_2, L_3, cutting the transversal T in the points Q, P, and M, making segment m equal to segment n; also cutting the transversal T_1 at the points B, A, and C, forming the segments r and s.

We are to prove that segment r equals segment s.

Analysis. To prove that segments r and s are equal, we may use triangles containing r and s. In order that such triangles may be proved congruent they must also contain segments m and n or the equals of m and n. In order to accomplish the latter we draw the lines AK and BH through the points A and B respectively parallel to transversal T, forming the triangles CKA and AHB. These triangles can be proved congruent by proving that two angles and a side of one are equal respectively to two angles and the corresponding side of the other.

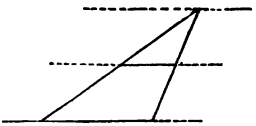
Proof.

What kind of a figure is $PHBQ$?	Why?
Then how does line BH compare with segment n?	Why?
What kind of a figure is $MKAP$?	Why?
How does line AK compare with segment m?	Why?

How do the segments m and n compare? Why?
What can you now say of lines BH and AK? Why?
How do angles ABH and CAK compare? Why?
How do angles HAB and KCA compare? Why?
What now can you say of ⊿ AHB and CKA? Why?
What can you state about the line segments r and s?

State the proposition proved.

84. Theorem XXX. *A line drawn through the mid-point of one side of a triangle parallel to a second side will bisect the third side.*

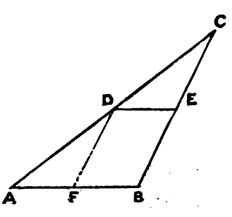


Suggestion. Through the vertex opposite the second side of the triangle draw a line parallel to the given line, and apply Theorem XXIX.

85. Theorem XXXI. *A line drawn through the mid-points of two sides of a triangle is parallel to the third side.*

Suggestion. Through one of the mid-points draw a line parallel to the third side. According to Theorem XXX this new line bisects the second side. This new line must coincide with the line which joined the mid-points of the two sides. Why? So that both are parallel to the third side. Give this proof in full.

86. Theorem XXXII. *The line joining the mid-points of two sides of a triangle is equal to one-half of the third side.*



Suggestion. D and E are mid-points of AC and BC. To prove $DE = \frac{1}{2} AB$, take F, the mid-point of AB and draw DF. From Theorem XXXI what can you say about lines DE and AB? About lines FD and BC? What kind of a figure is $BEDF$? What do you conclude about the lengths of DE and AB?

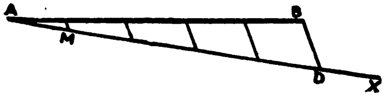
87. Problem XII. *To divide a given line-segment into n equal parts.*

Given the line segment AB.

To divide the line-segment AB into n equal parts.

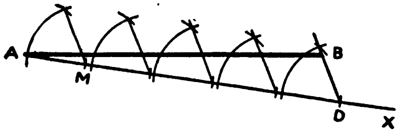
We shall let $n = 5$. The method is the same for any other number.

Analysis. Suppose the work done as in the figure below.



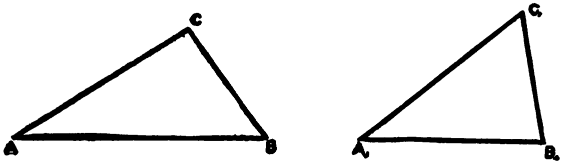
AX is a line of indefinite length making any acute angle with the given line AB. AM is a segment whose length is chosen at will. This length is laid off five times on the line AX, and the final point is marked D. BD is drawn. The lines through the other points on AX are drawn parallel to BD. These lines divide AB into five equal parts. Why?

Construction. At one end of the line-segment AB draw line AX making any acute angle with AB. Starting at point A lay off on AX any convenient length as AM as many times as the number of divisions you wish AB to contain. Call the last point D. Draw BD. Through each point on AX construct an angle equal to $\angle BDA$ making use of Problem III. Extend the arms of these angles to cut segment AB. Segment AB will be divided into the desired number of equal parts.



Proof. Let the student give the proof, using Theorem XXIX.

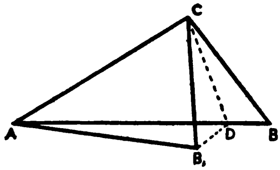
88. Theorem XXXIII. *If two triangles have two sides of the one respectively equal to two sides of the other, but the included angle of the one greater than the included angle of the other then the third sides are unequal, and the greater angle has the greater side opposite it.*



Given the triangles ABC and $A_1B_1C_1$ with side BC equal to side B_1C_1 and side CA equal to side C_1A_1, but angle ACB greater than the angle $A_1C_1B_1$.

To prove that side AB is greater than side A_1B_1.

Analysis. To prove this we shall place the two triangles together, placing $\triangle A_1B_1C_1$ on $\triangle ABC$, so that A_1C_1 falls on AC. Then C_1B_1 will fall between AC and CB. Why? If now we can so draw in a line that we shall have congruent triangles, and at the same time get a connection between AB and AB_1, we can compare the lengths of AB and AB_1. To do this we bisect $\angle B_1CB$ by the line CD and join B_1D.



We can now show that AB is greater than AB_1, if we can show that B_1D is equal to DB. We can show that BD is equal to DB, if we can show that $\triangle B_1DC$ is congruent to $\triangle DBC$. We can show that these two triangles are congruent.

Proof. $\triangle BCD \cong \triangle B_1DC$. Prove this.

$\therefore$ $DB = B_1D$. Why?

But $AD + DB_1 > AB_1$; Why?

$\therefore$ $AB > AB_1$. Why?

State the proposition proved.

Definition. When two theorems are related to one another in such a way that the thing that is to be proved in the one becomes the thing that is assumed to be true in the other, such theorems are called **converse** theorems. Either one is the converse of the other.

An example of converse theorems:

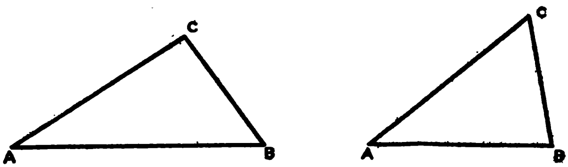
If two sides of a triangle are equal the angles opposite these sides are equal.

If two angles of a triangle are equal, the sides opposite these angles are equal.

State other propositions and their converses that you have proved.

The following theorem is the converse of Theorem XXXIII.

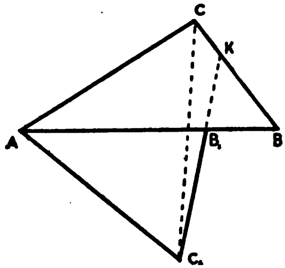
89. Theorem XXXIV. *If two triangles have two sides of the one equal, respectively, to the two sides of the other, but the third side of the one greater than the third side of the other, then the included angles are unequal, the greater side having the greater angle opposite it.*



Given the two triangles ABC **and** $A_1B_1C_1$ **with the side** BC **equal to side** B_1C_1 **and the side** CA **equal to side** C_1A_1**, but the side** AB **longer than** A_1B_1**.**

To prove that angle ACB is greater than angle $A_1C_1B_1$.

Analysis. To prove this, we place the two triangles together, placing A_1 on A, A_1B_1 taking the direction of AB, the two triangles on opposite sides of AB. Draw the line CC_1 and extend C_1B_1 to intersect BC at the point K.



Now since $\angle$s ACC_1 and CC_1A are equal, we can prove $\angle\ ACB$ greater than $\angle\ B_1C_1A$ if we can prove $\angle\ C_1CB$ greater than $\angle\ B_1C_1C$. We can prove $\angle\ C_1CB$ greater than $\angle\ B_1C_1C$, if we can prove that in $\triangle CC_1K$ side C_1K is greater than side CK. C_1K is greater than C_1B_1 which is equal to CB, which in turn is greater than CK.

Proof.	$\angle\ ACC_1 = \angle\ AC_1C.$	Why?
	$C_1K > C_1B_1.$	Why?
	$C_1B_1 = CB.$	Why?
	$CB > CK.$	Why?
$\therefore$	$C_1K > CK.$	
$\therefore$, in $\triangle\ C_1KC$,	$\angle\ C_1CK > \angle\ KC_1C.$	Why?
$\therefore$	$\angle\ ACB > \angle\ B_1C_1A$	Why?

State the proposition just proved.

Note. A proposition may be true and its converse not true.

For example:

If two angles are right angles, their sum will equal a straight angle.

The converse,

If the sum of two angles is a straight angle, the angles are right angles, is not true.

90. Exercises.

1. If the mid-points of the sides of an equilateral triangle are joined, four congruent equilateral triangles are formed. Prove.

2. Start with a given equilateral triangle (rather large); applying any of the preceding problems needed, divide it into four equal triangles.

3. So construct that the figure of Exercise 2 shall contain sixteen equilateral triangles.

4. If you should divide each of the equilateral triangles of Exercise 3 into four equilateral triangles, how many triangles would you have? By another such division how many triangles would you have? Write down in a row the numbers of triangles in Exercises 2, 3, 4. Do you find any common relation between these numbers? What is the relation? The figure which you have made is frequently used as a design for linoleum and tiling. See Exercise 26, page 52.

5. Construct six equilateral triangles about a point. Divide each into four equilateral triangles. How many triangles are there about each point of intersection of the lines?

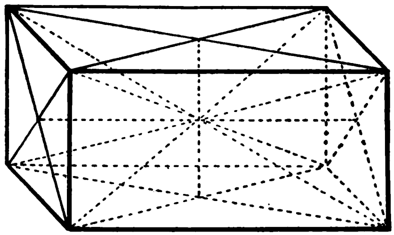
6. Make other designs by bringing together equilateral triangles of various sizes, and joining mid-points of sides.

7. Examine tiling floors, linoleums, laces and like forms to see if the equilateral triangle is brought often into use.

8. If a man sets five plants at equal distances apart and in a line with the mid-points of two of the border lines of a flower bed in the shape of an equilateral triangle, how many plants can he place the same distance apart around the border?

9. The mid-point of the hypotenuse of a right-angled triangle is equidistant from the three vertices. Prove. Could the mid-point of any side of an isosceles triangle be equidistant from the three vertices?

10. Take a card-board box and with needle and thread stretch lines from corner to corner, also from mid-point of side to mid-point of side, as shown in the figure.



Assuming that the lines which are parallel to the same straight line are parallel to each other no matter whether they are all in the same plane or not, prove the following statements:

The diagonals of each face bisect each other;

The diagonals of the figure bisect each other;

The lines joining the mid-points of the faces bisect each other, and are parallel to the edges.

11. Take a card-board box and take the bottom out, and shear it to the side. Treat as in Exercise 10. State the facts that are the same as before, and the variations that have been made.

12. If the sides of a quadrilateral are bisected, the lines joining the points of bisection in order form a parallelogram.

Hint. Draw in the diagonals of the quadrilateral.

Note. When a quadrilateral is called for, be careful not to draw a special figure such as a parallelogram or square.

13. The lines which join the mid-points of the opposite sides of a quadrilateral bisect each other.

14. Two straight lines are cut by three parallel lines. Call the segments which are cut on one of the transversals a and b respectively and the segments cut on the other c and d respectively. If segment a contained 6 units less, it would be

$\frac{1}{2}$ as long as segment *c*. If 8 units were added to twice segment *b*, it would be just twice segment *d*. If segment *a* equals segment *b*, find the lengths of segments *a*, *b*, *c*, and *d*.

15. If, in Exercise 14, segment *a* contained *m* units less it would be $\frac{p}{q}$ part of segment *c*. If *r* units were added to *n* times segment *b*, it would be *r* times segment *d*. If segment *a* equals segment *b*, find the lengths of segments *a*, *b*, *c*, and *d*.

91. Summary of Chapter II.

Part I. Angles. Measurement—Degree and Radian.

Postulate V. All straight angles are equal.

Corollary. All right angles are equal.

Complementary, Supplementary, Conjugate Angles.

Theorem I. If two angles are equal, their complements are equal.

Theorem II. If two angles are equal, their supplements are equal.

Theorem III. If two angles are equal, their conjugates are equal.

Vertical Angles.

Theorem IV. If two straight lines intersect, the vertical angles are equal.

Part II. Triangles—Congruence and Construction.

Congruent Triangles.

Theorem V. If two triangles have two sides and the included angle of one respectively equal to the sides and the included angle of the other, the triangles are congruent.

Theorem VI. If two triangles have two angles and the included side of the one respectively equal to the two angles and the included side of the other, the triangles are congruent.

Isosceles Triangles. Equilateral Triangles. Definition.

Theorem VII. In an isosceles triangle the angles opposite the equal sides are equal.

Corollary. If a triangle is equilateral it is also equiangular.

Perpendicular lines. Definition.

Corollary. At a point in a line only one perpendicular can be erected to that line.

Theorem VIII. Every point in the perpendicular bisector of a straight line is equidistant from the ends of the line.

Theorem IX. If two triangles have three sides of the one equal respectively to the three sides of the other, the triangles are congruent.

Constructions.

Circumference. Assumptions with regard to circumferences.

Problem I. To construct a triangle when three sides are given.

Problem II. To construct a triangle congruent to a given triangle. This is another form of Problem I.

Theorem X. The sum of two sides of a triangle must be greater than third side and their difference less than the third side.

Problem III. To bisect a given angle.

Corollary 1. At a given point on a given line to construct a perpendicular to the given line.

Corollary 2. From a given point external to a given line to construct a perpendicular to the given line.

Problem IV. To construct the perpendicular bisector of a given line-segment.

Problem V. At a given point on a given line to construct an angle equal to a given angle.

Problem VI. To construct a triangle when two sides and the included angle are given.

Problem VII. To construct a triangle when two angles and the included side are given.

Exterior angles of triangles and polygons. Definition.

Theorem XI. The exterior angle of a triangle is greater than either interior angle not adjacent to it.

Theorem XII. If two sides of a triangle are unequal, the angles opposite are unequal, and the greater side has the greater angle opposite it.

Part III. Transversals—Parallel lines—Sums of angles of polygons—Parallelograms.

Definitions. Alternate interior angles, alternate exterior angles, corresponding angles.

Theorem XIII. If two straight lines are cut by a transversal, making one pair of alternate interior angles equal, the other pair of alternate interior angles are equal, the pairs of alternate exterior angles are equal, the pairs of corresponding angles are equal, the sum of the two interior angles on the same side of the transversal is equal to a straight angle, and the sum of the exterior angle on the same side of the transversal is equal to a straight angle.

Corollary. As above starting with pairs of exterior angles equal.

Definition. Parallel lines.

Postulate of Parallels. Two intersecting lines cannot be parallel to the same line.

Theorem XIV. If two straight lines are cut by a transversal making a pair of alternate interior angles equal the lines are parallel.

Problem VIII. Through a given point to construct a line parallel to a given line. Involved in Theorem XIII, XIV.

Converse Propositions. Illustrations and definition.

Theorem XV. If two parallel lines are cut by a transversal the alternate interior angles are equal, etc.

Theorem XVI. A line perpendicular to one of two parallel lines is perpendicular to the other.

Theorem XVII. A line parallel to one of two parallel lines is parallel to the other.

Theorem XVIII. Two angles whose arms are parallel each to each are either equal or supplemental.

Theorem XIX. Two angles whose arms are perpendicular each to each are either equal or supplemental.

Theorem XX. The sum of the angles of a triangle is a straight angle.

Corollary 1. The exterior angle of a triangle is equal to the sum of the interior angles not adjacent to it.

Corollary 2. A triangle can have but one right angle or one obtuse angle.

Corollary 3. If two triangles have two angles of the one equal respectively to two angles of the other, the third angles are equal.

Corollary 4. If two triangles have two angles and a side of one equal respectively to two angles and the corresponding side of the other, the triangles are congruent.

Corollary 5. If two right triangles have an angle and a side of one equal respectively to an angle and the corresponding side of the other the triangles are congruent.

Theorem XXI. If two angles of a triangle are equal the sides opposite those angles are equal.

Problem IX. Given two angles of a triangle, to construct the third angle.

Problem X. To construct a triangle, given two angles and a side opposite one of them.

Problem XI. To construct a triangle, given two sides and an angle opposite one of them.

Theorem XXII. If two angles of a triangle are unequal the sides opposite those angles are unequal and the greater angle has the greater side opposite it.

Corollary 1. Of all lines drawn to a given straight line from a given external point, the perpendicular is the shortest.

Corollary 2. The hypotenuse of a right angled triangle is the longest side of the triangle.

Corollary 3. Of two oblique lines drawn to a given line from a given external point that which cuts off the greater distance from the foot of the perpendicular is the longer.

Theorem XXIII. The sum of the interior angles of a polygon of n sides is $(n - 2)$ straight angles.

Theorem XXIV. The sum of the exterior angles of any polygon is 2 straight angles.

Parallelograms.

Definitions.—Parallelogram, rectangle, square, rhombus, diagonal.

Theorem XXV. If a convex quadrilateral has each pair of its opposite sides equal the figure is a parallelogram.

Theorem XXVI. If a convex quadrilateral has one pair of opposite sides equal and parallel, the figure is a parallelogram.

Theorem XXVII. A diagonal divides a parallelogram into two congruent triangles.

Corollary 1. The opposite sides of a parallelogram are equal.

Corollary 2. The opposite angles of a parallelogram are equal.

Corollary 3. The sum of any two consecutive angles of a parallelogram is equal to a straight angle.

Theorem XXVIII. The diagonals of a parallelogram bisect each other.

More than two parallel lines cut by transversal.

Theorem XXIX. If a system of parallels cuts equal parts on any transversal, it does on every transversal.

Theorem XXX. If a line is drawn through the mid-point of one side of a triangle, parallel to a second side, it bisects the third side.

Theorem XXXI. If a line is drawn through the mid-points of two sides of a triangle, it is parallel to the third side.

Theorem XXXII. If a line is drawn through the mid-points of two sides of a triangle, it is equal to one-half of the third side.

Problem XII. To divide a line-segment into n equal parts.

Theorem XXXIII. If two triangles have two sides of the one respectively equal to two sides of the other, but the included angles unequal, then the third sides are unequal, the greater side being opposite the greater angle.

Theorem XXXIV. If two triangles have two sides of the one respectively equal to two sides of the other, but the third sides unequal, then the included angles are unequal, the greater angle being opposite the greater side.

1. Into what three parts is this chapter divided as to subject matter?
2. Are the subjects entirely independent?
3. Read carefully the theorems of Part II, and state those (if any)

which were proved independently of Part I. State those (if any) which depend for their proof on theorems in Part I.

4. Go through the theorems of Part III, and state those that depend on the theorems in Part I, then those that depend on the theorems in Part II, then those that depend on theorems from both of these.

5. What two things are involved in the statement of a theorem?

6. What directs you in the drawing of a figure?

7. What directs you as to what you shall state is given? As to what you shall state is to be proved?

8. What is the object of an analysis?

9. In general how do the steps in the proof correspond to the steps in the analysis?

10. Read over the list of exercises and pick out those that seem to you to have real application, that is, seem directly related to real things around you.

In the history at the opening of this book it was stated that the geometry which Euclid wrote, 300 B. C., was divided into thirteen books. The theorems of this chapter are selected from Book I.

CHAPTER III

EQUALITY OF FIGURES

92. Equal Figures.* If you draw a geometric figure of any shape, cut it up and put the parts together in a different order, the new figure will differ from the old in form, but not in size or area.

Definition. Figures having the same size are called **equal figures.**

Addition and subtraction of polygons. **Two convex polygons** may be added by placing one without the other with a portion of their perimeters together and then erasing that portion of their perimeters. The new polygon will be the sum. Two convex polygons may be subtracted by placing one within the other. The difference between their areas can readily be decided.

A polygon, such as a star-shaped figure, can easily be divided, if necessary, into several convex polygons by drawing in one or more lines.

93. Definitions.

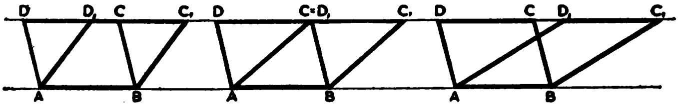
Either set of parallel sides of a parallelogram may be regarded as the **bases,** in which case the perpendicular distance between the two parallel lines is the corresponding **altitude.**

Any side of a triangle may be regarded as its **base,** then the perpendicular from the opposite vertex to the chosen side is the corresponding **altitude.**

The parallel sides of a trapezoid are its **bases,** and the perpendicular distance between the two bases is the **altitude.**

* **In connection with this chapter, First Course, pages 136–144 may be reviewed.**

94. Theorem I. *Parallelograms on the same base, or on equal bases, and between the same parallel lines are equal*



Given the parallelograms $ABCD$ and ABC_1D_1 standing on the same base AB and between the same parallel lines.

To prove the parallelogram ABCD equal to the parallelogram ABC_1D_1.

Note. The three figures show the three ways in which the parallelograms can stand with reference to each other.

Analysis. ▱s $ABCD$ and ABC_1D_1 are not the same shape, but we can show them to be equal if we can show that for every portion of the one there is an equal portion of the other. In the first figure parallelogram $ABCD$ is divided into the two portions, triangle AD_1D and the trapezoid $ABCD_1$, while parallelogram ABC_1D_1 is divided into triangle BC_1C and trapezoid $ABCD_1$. Since the trapezoid $ABCD_1$ is common to both, we have but to prove that $\triangle AD_1D$ is congruent to $\triangle BC_1C$. $\triangle AD_1D$ is congruent to $\triangle BC_1C$ if we can prove that AD equals BC, that AD_1 equals BC_1 and that included angle D_1AD equals included angle C_1BC.

Proof. In the ▵s AD_1D and BC_1C

$AD = BC$. Why?

$AD_1 = BC_1$. Why?

$\angle D_1AD = \angle C_1BC$. (Art. **62**) Why?

$\therefore$ $\triangle AD_1D \cong \triangle BC_1C$. Why?

$\therefore$ $\triangle AD_1D +$ trapezoid $ABCD_1$

$= \triangle BC_1C +$ trapezoid $ABCD_1$. Why?

$\therefore$ ▱ $ABCD$ = ▱ ABC_1D_1. Why?

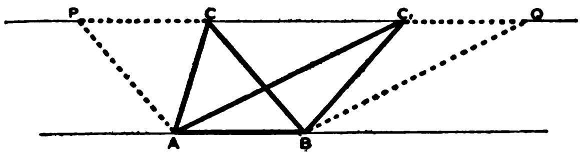
Give analysis and proof for each of the other figures.

State the proposition proved.

Corollary 1. *Parallelograms having equal bases and equal altitudes are equal.*

Corollary 2. *A parallelogram equals a rectangle which has the same base and the same altitude.*

95. Theorem II. *Triangles which stand on the same base and are between the same parallel lines are equal.*



Given the triangles ABC and ABC_1 standing on the same base and between the same parallel lines.

To prove triangle ABC equal to triangle ABC_1.

Analysis. Since we know that parallelograms are equal if they have the same bases and are between the same parallel lines, we can show that ⩟ ABC and ABC_1 are equal if we can show that they are halves of such parallelograms. To do this draw AP parallel to side BC, and BQ parallel to side AC_1 forming ▱s $ABCP$ and $ABQC_1$ respectively. Then show that △ ABC is one-half of ▱ $ABCP$, and that △ ABC_1 is one-half of ▱ $ABQC_1$.

Proof. Let the student give this proof.

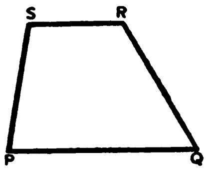
Corollary 1. *Triangles which have equal bases and equal altitudes are equal.*

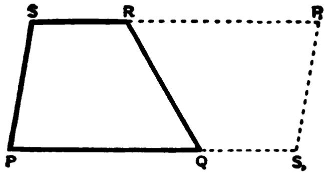
Corollary 2. *A triangle equals half a parallelogram which stands on the same base and lies between the same parallel lines.* This might be stated.

Corollary 3. *A triangle is equal to one-half of a parallelogram which has the same base and altitude.*

Exercise. A parallelogram is divided into four triangles by drawing in the diagonals. Compare their areas.

96. Theorem III. *A trapezoid is equal to one-half of a rectangle whose base is the sum of the parallel sides and whose altitude is the same as the altitude of the trapezoid.*





Given the trapezoid *PQRS*.

To prove that the trapezoid PQRS is equal to one-half of a rectangle whose base is equal to PQ plus RS and whose altitude is the same as that of the trapezoid.

Analysis. We shall first prove that the trapezoid *PQRS* is equal to half the parallelogram whose bases are the sum of *PQ* and *RS*, and whose altitude is the same as that of the trapezoid. In order to have such a parallelogram, take a trapezoid identical to trapezoid *PQRS*, invert it and place it so that side *QR* falls on side *RQ* as shown in the figure.

Call the new figure PS_1P_1S. Figure PS_1P_1S will be the required parallelogram provided that we can prove that QS_1 falls in a straight line with *PQ* and that RP_1 falls in a straight line with *SR*, and that P_1S_1 is equal and parallel to *SP*. We can prove that QS_1 falls in a straight line with *PQ* if we can prove that $\angle\ RQP + \angle\ S_1QR$ equals a straight angle, and we can prove that line RP_1 falls in a line with *SR* if we can prove that $\angle\ SRQ + \angle\ QRP_1$ equals a straight angle. Since trapezoid QS_1P_1R is the trapezoid inverted, this amounts to proving that $\angle\ SRQ + \angle\ RQP$ is equal to a straight angle.

Proof. $\angle SRQ + \angle RQP$ = a straight angle. Why?

$\angle S_1QR = \angle SRQ$. Why?

∴ $\angle S_1QR + \angle RQP$ = a straight angle. Why?

∴ PQ and QS_1 are in a straight line. Why?

In like manner RP_1 is in a straight line with SR.

$PS_1 \parallel SP_1$. Why?

$PS_1 = SP_1$. Why?

∴ PS_1P_1S is a parallelogram. Why?

∴ Trapezoid $PQRS$ is one-half of parallelogram PS_1P_1S. Why?

Therefore trapezoid $PQRS$ is equal to one-half of a rectangle whose base is $PQ + RS$ and whose altitude is the same as that of the trapezoid. Why?

State the proposition proved.

97. From experiments, as in First Course, page 22, § 16, it can easily be brought out that the number of square units in the area of a rectangle is equal to the number of linear units in the base multiplied by the number of linear units in the altitude. Briefly stated, *the area of a rectangle equals the product of its base by its altitude.*

For the present we shall assume this. A formal proof will be given later.

Corollary 1. *The area of a parallelogram is equal to the product of its base by its altitude.* (It must be remembered in these statements that it is *numbers* that are equal, namely, the *number* of square units of area is equal to the *product* of the *number* of linear units in the base by the *number* of linear units in the altitude.)

Corollary 2. *The area of a triangle is equal to one-half of the product of the base by the altitude.*

Corollary 3. *The area of a square is equal to the square of one of its sides.*

Corollary 4. *The area of a trapezoid is equal to one-half of the product of the sum of its bases by the altitude.*

The above may be expressed algebraically,

for the rectangle	$a = bh,$
for the parallelogram	$a = bh,$
for the triangle	$a = \frac{1}{2}bh,$
for the trapezoid	$a = \frac{1}{2}(b_1 + b_2)h,$

where a, b and h equal the number of units in the area, base, and altitude of the figures respectively.

98. Exercises. Algebraic and Geometric

1. The length of a rug is 5 feet and the width is $3\frac{1}{2}$ feet. What is the number of square feet in its area?

2. The design of the rug spoken of in Exercise 1 is a border $\frac{1}{2}$ foot wide about a plain center. Find the area of the center and also the border.

3. It takes 103 square feet and 128 square inches of linoleum to cover a hallway. The hall is 6 feet 8 inches wide. How long is it?

4. If a rectangle is $\frac{3}{4}ab$ square units in area, and $\frac{2}{3}a$ units in width, how long is it?

5. If a parallelogram is $\frac{a}{b} + \frac{c}{d}$ square units in area, and $\frac{a}{b} - \frac{c}{d}$ units in width, how long is it?

Note. If the student finds that he cannot handle his fractions readily he should look up the work in a text-book on Algebra, as First Course, pages 150 to 156 inclusive.

6. If a triangle is $\frac{3}{5}a + \frac{2}{7}b$ square units in area and $\frac{1}{5}a - \frac{3}{35}b$ units in altitude, what is the number of units in the base?

7. A trapezoid has an area of $1\frac{1}{2}r - 2\frac{3}{4}s$ square units. One of its bases is $1\frac{2}{5}r + \frac{5}{14}s - 3$ and the other base is $\frac{2}{3}r - \frac{9}{14}s + 3$ units. Find the altitude.

8. A cut through a hill is 25 feet wide at the top and 12 feet wide at the bottom. It has a depth of 10 feet. What is the area of a cross section?

9. Measure the width and height, including the sloping roof, of a bird house and compute the area of a cross section.

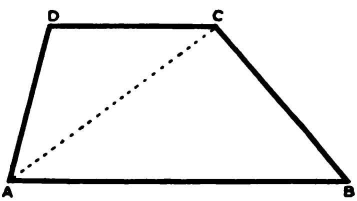
10. Measure the distance across the top and bottom, and the height of a cork. Compute the area of a section through the axis of the cork.

11. Points P and Q are any points in the sides AB and CD respectively of ▱ $ABCD$. The straight lines PD, PC, QA, and QB are drawn. Prove that $\triangle\ PCD$ equals $\triangle\ ABQ$. Give both an algebraic and a geometric proof.

12. A line is drawn parallel to two sides of a parallelogram, and cutting the other two sides. A point P on the line is joined to the vertices of the parallelogram. Prove that the sum of the two triangles formed with the first two sides of the parallelogram is equal to one-half the area of the parallelogram. Give both algebraic and geometric proof.

13. P is any point in the parallelogram $MNHK$. Lines are drawn joining P to each of the vertices of the parallelogram. Prove that the sum of △s MPK and PNH equals the sum of △s MNP and PHK. Give both algebraic and geometric proof.

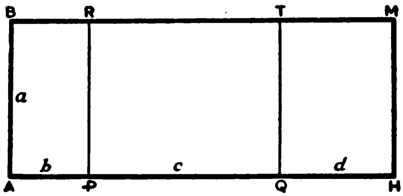
14. Prove Corollary 4 by means of the adjacent figure.



Note. In connection with the following it is advisable to review First Course, pages 127 to 135 inclusive.

99. To shorten the naming of a rectangle it may be named diagonally across. For the same reason we shall adopt the plan of naming by the lengths of two adjacent sides. There is a double convenience in this, since the product of these two lengths gives the area.

100. Theorem IV. *The rectangle of two given lines equals the sum of the rectangles contained by one of them and the several segments into which the other is divided.*



Given the rectangle on the lines AB and AH with line AH divided into the segments b, c, d by the points P and Q. PR and QT are lines drawn parallel to AB. Line AB is a and line AH is $b + c + d$.

To prove that $a(b + c + d) = ab + ac + ad$.

Analysis. Since rectangle $a\ (b + c + d)$ is made up of ▭ $APRB$, $PQTR$ and $QHMT$, we have but to prove that PR and QT are each equal to a, so that ▭ $APRB$ = ▭ ab, $PQTR$ = ▭ ac, and ▭ $QHMT$ = ▭ ad.

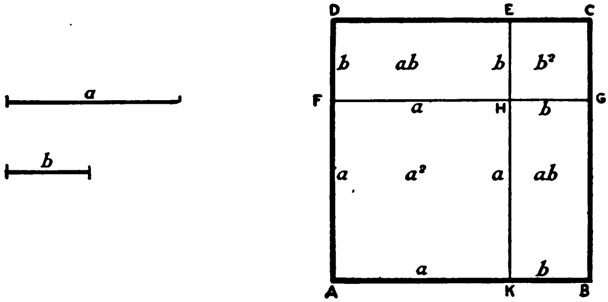
Proof. Let the student give this proof in full.

101. Exercises. Write the dimensions of the rectangles which are composed of the following sums:

1. $mh + mk + mr$.
2. $p^2 + pq + 2p$.
3. $t^2 + 3t + ts$.
4. $4r + 28$.
5. $25 + 30 + 45$.
6. $2ms + 4rs + 6s$.

Notice that we have here simply the algebraic process of factoring, illustrated geometrically. Thus the rectangle whose area is $3hm + 9hn + 6hq$ can be regarded as a sum of three rectangles, namely $3hm$, $9hn$, and $6hq$. These can be regarded as having a common dimension $3h$, thus: $3h \cdot m$, $3h \cdot 3n$, and $3h \cdot 2q$. Their sum then is a rectangle $3h \cdot (m + 3n + 2q)$, which is the factored form of the original expression.

102. Theorem V. *The square on the sum of two line segments is equal to the sum of the squares on these segments plus two times their rectangle.*



Given the line segments a **and** b**, and the square** $ABCD$ **on the sum of these two lines.**

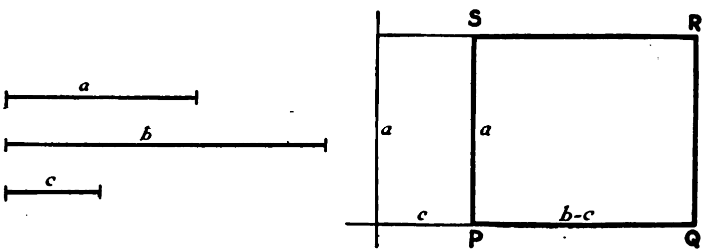
To prove that $(a+b)^2 = a^2 + 2ab + b^2$.

Draw KE parallel to AD, and FG parallel to AB, cutting KE in point H. Analyze and prove.

103. Exercises. Write the side of the squares which have the following areas:

1. $m^2 + 4m + 4$.
2. $9p^2 + 12pq + 16q^2$.
3. $25 + 40 + 16$.
4. $225 + 30r + r^2$.
5. $625 + 100p + 4p^2$.
6. $49s^2 + 42ps + 9p^2$.

104. Theorem VI. *The rectangle of a given line and the difference between two other given lines is equal to the difference between the rectangles of the one line and each of the others.*



Given a, b, c the length of three lines, and the rectangle of a and $(b - c)$.

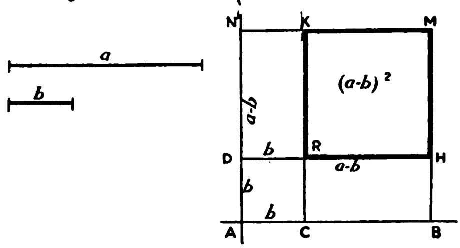
To prove that $a(b - c) = ab - ac$.

Let the student give analysis and proof.

105. Exercises. Write the dimensions of the rectangle which is the difference between the following rectangles.

1. $2a - 2c$.
2. $5r - 25$.
3. $b^2 - bc$.
4. $p^2 - pq$.
5. $2hk - 8k^2 - k$.
6. $3s^2 - 27rs + 9st$.

106. Theorem VII. *The square on the difference between two lines equals the sum of the squares on the two lines minus two times their rectangle.*



Given the lines a and b, with $RHMK$ the square on their difference.

If line AB is a and line AC is b, then the square $RHMK$ is $(a - b)^2$.

To prove that $(a - b)^2 = a^2 + b^2 - 2ab$.

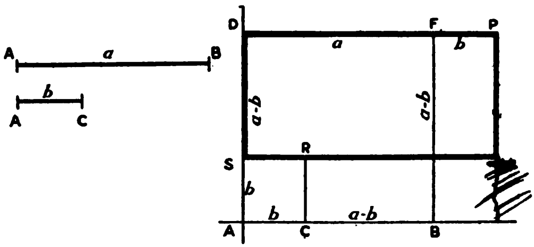
Analysis. To prove that $(a - b)^2 = a^2 + b^2 - 2ab$, we prove that $ABMN$ is the square on a, that ▭ $ABHD$ which is subtracted from $ABMN$ is ▭ ab, that $ACRD$ is the square on b and must be added before we can subtract ▭ $ACKN$, which is ▭ ab. Thus we are to show that $RHMK$ is the result of subtracting from a^2 the rectangle ab, then adding b^2, and then subtracting rectangle ab. This will prove our statement.

Proof. Let the student give this proof.

107. Exercises. Write the side of the square whose area is as follows:

1. $9 - 12 + 4$.
2. $16 r^2 - 40 rs + 35 s^2$.
3. $81 p^2 - 18 p + 1$.
4. $t^2 - 30 ts + 225 s^2$.
5. $121 b^2 - 66 b + 9$.
6. $49 - 70 + 25$.

108. Theorem VIII. *The difference of the squares of two lines is equal to the rectangle of the sum and difference of the lines.*



Given the segments $AB = a$, $AC = b$, SD their difference and SM their sum. Then rectangle $SMPD$ is the rectangle of $(a + b)(a - b)$.

To prove that $a^2 - b^2 = (a + b)(a - b)$.

Analysis. Since $ABFD$ is a^2 and $ACRS$ is b^2, then the irregular figure $SRCBFD$ is the difference between a^2 and b^2. We must prove that ▭ $SMPD$ is equal to irregular figure $SRCBFD$.

Proof. Let the student give this proof in full.

109. Exercises. Write the dimensions of the rectangles equal to the following differences of squares.

1. $16 - 9$.
2. $25 a^2 - 36$.
3. $144 m^2 - n^2$.
4. $121 d^2 - 49 c^2$.
5. $4 h^2 - 36$.
6. $36 - 1$.
7. By the same plan of proof as given for Theorem V prove that

$$a^2 + am + an + mn = (a + m)(a + n).$$

Check by substituting numbers for a, m, and n.

8. Write the dimensions of the rectangle composed of $s^2 + 5s + 6$.

9. By the same plan of proof as given in Theorem VII prove that

$$a^2 - am - an + mn = (a - m)(a - n).$$

10. By the same plan prove that

$$a^2 + am - an - mn = (a - n)(a + m).$$

11. By drawing, find the side of the rectangle whose area is expressed by

$$a^2 + 2ab + b^2 - m^2 + 2mn - n^2.$$

Suggestion: Note that the given expression is equal to

$$(a^2 + 2ab + b^2) - (m^2 - 2mn + n^2),$$

which is equal to

$$(a + b)^2 - (m - n)^2.$$

To this apply Theorem VIII, page 94.

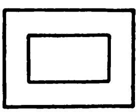
Check by substituting numbers for a, b, m, n.

12. Write the dimensions of the following rectangles:

(a) $r^2 - 7rs + 10s^2$.
(b) $k^2 - 2k - 8$.
(c) $6m^2 - 5m - 1$.
(d) $1 - p^2 - 2pr - r^2$.
(e) $3a^2 + 10a + 3$.
(f) $5r^2 - 7r + 2$.
(g) $4m^2 - 5m - 9$.
(h) $7p^2 + 11ps - 6s^2$.

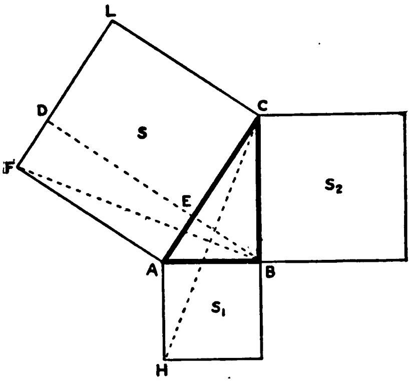
13. It required 90 square blocks of tiling to make a floor for a hallway. It was necessary to use 3 more than twice as many in the length as in the width. If each block was a foot square, how long and how wide was the hall?

14. Plants are arranged in the form of a hollow rectangle at uniform distance apart. Along the outer row there are 18 one way and 11 the other. How many rows will it take for 138 plants?



15. In a strip of wall-paper (design of flowers on a plain ground) there is one more than 3 times as many in the length as in the width. There were 14 flowers on the strip. How long is the strip if the flowers are 8 inches apart?

110. Theorem IX. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.*



Given the right triangle ABC with B the right angle, and S, S_1, S_2 the squares on the hypotenuse and sides respectively.

To prove that $S = S_1 + S_2$.

Analysis. To make this proof we divide the square S into two parts by drawing a line BD through B parallel to AF, the side of square S. If we can now prove that ▭ $AEDF$ is equal to square S_1, and that ▭ $ECLD$ is equal to square S_2, we shall have proved our theorem. Consider ▭ $FAED$ and square S_1 first. We can prove these equal if we make two triangles, one with a base and altitude equal to those of ▭ $FAED$, the other with base and altitude equal to those of square S_1, and then prove these triangles congruent. In order to have these triangles draw the lines BF and CH.

Proof. In ⊿ ABF and AHC,

$AF = AC$. Why?

$AB = AH$. Why?

$\angle BAF = \angle HAC$. Why?

(Notice of what two parts each angle is the sum.)

$\therefore \quad \triangle FAB \cong \triangle AHC.$ Why?

$\square FAED = 2 \triangle FAB.$ Why?

Square $S_1 = 2 \triangle AHC.$ Why?

(Explain why CB and HA are parallel.)

$\therefore \quad \square FAED =$ square S_1. Why?

Let the student give a complete proof that

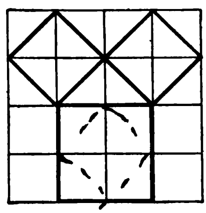
$\square ECLD =$ square S_2.

$\therefore \quad \square FAED + \square ECLD =$ square $S_1 +$ square S_2. Why?

$\therefore \quad$ square $S =$ square $S_1 +$ square S_2.

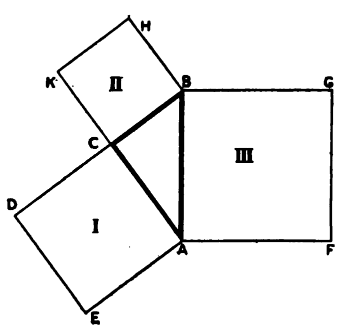
State the proposition proved.

This theorem is called the **Pythagorean Theorem** from Pythagoras (580–500 B. C.) who first proved it. Many proofs have been given since his time. The one given here is attributed to Euclid (about 300 B. C.). Pythagoras used the adjacent figure for his proof. It is supposed to have been suggested to him by the pattern in a tile floor. This theorem was known many centuries before the time of Pythagoras, but it had not been proved.

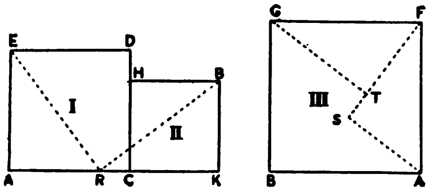


Is this figure general? Give a proof using this figure.

111. Prove the Pythagorean Theorem using the following figures. Here $\triangle ABC$ is shown with the squares on its three sides. In the second figure squares I and II are shown side by side, with dotted lines to a point R so placed that AR equals the side BC of $\triangle ABC$. The third figure shows square III, with AS and SF drawn in so as to make $\triangle AFS \cong \triangle ABC$



of the first figure. Then GT is drawn $\perp FS$.



Draw the second figure, taking $\triangle ABC$ in the first figure with sides $AC = 2$ inches, $BC = 1.5$ inches, $AB = 2.5$ inches. Cut your drawing into three parts by cuts along the dotted lines.

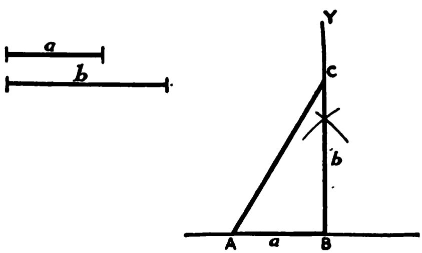
Draw the third figure, using same dimensions as above.

Show that the three parts into which you cut the second figure will exactly cover the third figure.

$\therefore$ square I + square II = square III.

See also First Course, pages 170–171.

112. Problem I. *To construct a right triangle when the two sides are given.*



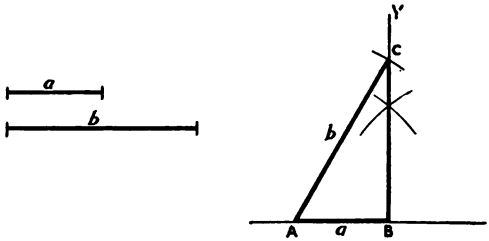
Given the segments a **and** b.

To construct a right triangle with a and b as sides.

Construction. Draw $AB = a$. At B, draw $BY \perp AB$. On BY lay off $BC = b$. Draw AC. Then $\triangle ABC$ is the required triangle.

Proof. Let the student give the proof, first making an accurate construction.

113. Problem II. *To construct a right triangle when one side and the hypotenuse are given.*



Given the side equal to a and the hypotenuse equal to b.

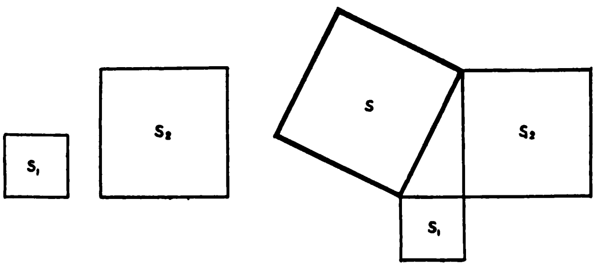
To construct a right triangle.

Construction. Draw $AB = a$, and at B draw $BY \perp AB$. With A as center and compass opened to the distance b, strike an arc cutting BY at C. Then $\triangle ABC$ is the required triangle.

Proof. Student supply proof.

Discussion. When will such a construction be impossible?

114. Problem III. *To construct a square which will be the sum of two given squares.*



Given the squares S_1 and S_2.

To construct a square equal to the sum of S_1 and S_2.

Let the student give a description of this construction, with reasons.

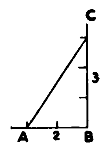
115. Problem IV. *To construct the square root of a number.*

We shall suppose the number to be a positive integer. If it is a perfect square, the construction can be made at once.

When the number is not a perfect square, we first express it as the sum of two or more squares, and then use the Pythagorean Theorem.

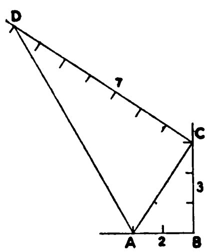
Example 1. Construct a line-segment of length $\sqrt{13}$.

$$13 = 2^2 + 3^2.$$



Take any unit of length.
Draw $AB = 2$ units.
Draw $BC = 3$ units, and $\perp AB$.
Then $AC = \sqrt{13}$ units. Why?

Example 2. Construct a line-segment of length $\sqrt{62}$.



$62 = 2^2 + 3^2 + 7^2$.
Take any unit of length.
Draw $AB = 2$ units.
Draw $BC = 3$ units, and $\perp AB$.
Then $AC = \sqrt{13}$ units.
Draw $CD = 7$ units, and $\perp AC$.
Then $AD = \sqrt{62}$ units.

Example 3. Construct a line-segment of length $\sqrt{24}$.

(*a*) Let $24 = 4^2 + 2^2 + 2^2$, and construct as in Example 2.

(*b*) Let $\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$. Construct $\sqrt{6}$, and double the result.

(*c*) Let $24 = 5^2 - 1^2$.
Construct a right triangle with a side 1 unit long, and hypotenuse 5 units long. See Art. **113.** The other side of this triangle will equal $\sqrt{24}$ units. Explain.

116. Exercises. Algebraic and Geometric.

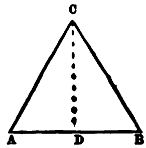
1. Find the side of a square which equals the sum of two given squares whose sides are 4 and 7 respectively; 13 and 8.

2. Construct a square which is twice the size of a given square.

3. What is the ratio between the side of the resultant square and the given square in Exercise 2?

4. Construct a square whose area is 23 square units; 37 square units; 8 square units; 52 square units. Note that $\sqrt{52} = \sqrt{4.13} = 2\sqrt{13}$. Construct $\sqrt{13}$ and double the result.

5. Find the altitude of an equilateral triangle whose side is 1 unit; whose side is 6 units; whose side is 25 units; whose side is s units. See illustration which follows.



Solution. Given the equilateral triangle ABC with side AB 1 unit in length.

To find the length of the altitude CD.

Analysis. Since ABC is an equilateral triangle, the altitude CD bisects the base, that is, AD is half the side AB. (Why?) Then ADC is a right-angled triangle with hypotenuse AC one unit in length, and side AD half a unit in length. Our problem is to find the length of side CD.

Let h = the number of units in CD.

Then $(\frac{1}{2})^2 + h^2 = 1^2$, by the Pythagorean Theorem.

$\therefore$ $h^2 = \frac{3}{4}$,

$\therefore$ $h = \pm \frac{1}{2}\sqrt{3}$, number of units in altitude.

Solve each of the other parts of the exercise in the same way.

The answer to the last, when the side is s units long, is

$$\mathbf{h} = \tfrac{1}{2}\mathbf{s}\sqrt{3}.$$

The student should fix this in mind as a formula. It will be needed many times in the future work.

6. Using the formula in Exercise 5 find the altitude of the equilateral triangles whose sides are 7; 32; 280; $a + b$; $4m - 10n$; $\frac{3}{4}r + \frac{1}{2}s - \frac{3}{4}$.

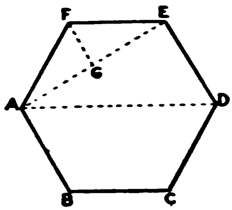
7. If the altitude of an equilateral triangle is h, find the length of its side.

8. Using this as a formula find the length of the side of the equilateral triangle whose altitude is 6; 11; 27; $3m - 2n$.

9. Find the areas of several of the triangles given in Exercises 5, 6, 7, 8.

10. Compute the area of a regular hexagon the length of whose side is 1 unit.

Suggestion. In the adjoining figure AD and AE are diagonals. FG is a line perpendicular to AE. Let the student show that $\angle GFE$ is $60°$, $\angle FEG$ is $30°$, and $\angle AED$ is $90°$. Then by Art. 72 Ex. 5, line FG is $\frac{1}{2}$ FE. From this find the length of diagonal AE and then AD. From this compute the area of the hexagon.

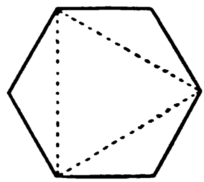


11. Compute the area of the regular hexagon whose side is 10 units in length.

12. Compute the area of a regular hexagon whose side is s units in length.

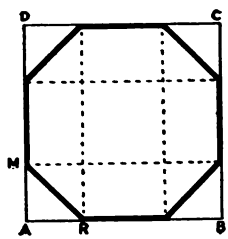
13. How does the area of a regular hexagon whose side is s units compare with the area of an equilateral triangle whose side is s units? (See Ex. 14, § 76.)

14. Compute the area of a regular hexagon by means of the adjoining figure. Compare with Exercise 10.



15. In the adjoining figure $ABCD$ is a square whose side AB is given. It is desired to cut off the corners so as to form a regular octagon.

Suggestion. The triangle ARM is an isosceles right triangle. We must find the length of MR in terms of the length of AB which we shall assume to be s.



Let d = the number of units in length of AR.

Then $d\sqrt{2}$ = the number of units in length of MR. Why?

Then $2d+d\sqrt{2}$ = the number of units in length of AB. Why?

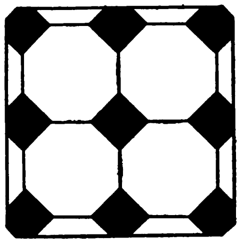
$\therefore$ $2d+d\sqrt{2} = s$. Why?

$$d = \frac{s}{2+\sqrt{2}}$$ (Rationalize denominator of fraction.)

$$d = (1 - \tfrac{1}{2}\sqrt{2})\, s, \quad \text{number of units in } AR.$$
$$d\sqrt{2} = \sqrt{(2} - 1)\, s, \quad \text{number of units in } MR.$$

Starting with a square construct a regular octagon. Do not use approximate values. Construct $\sqrt{2}\, s$ and take away s from it. The remainder is MR.

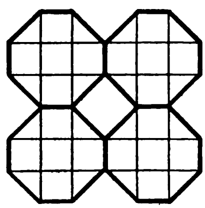
16. Suppose that the figure in Ex. 15 is a regular octagon the length of whose side is 10 units. By computing the areas into which it is divided obtain the area of the whole.



17. The adjoining figure may be used as a design for tiling. What is the shape of the dark blocks, if the other blocks are regular octagons? Prove your answer.

18. With regular octagonal blocks 6 inches on the side it takes 10 blocks one way and 12 blocks the other, together with a sufficient number of smaller blocks to fill in, to cover a floor. What is the area of the floor?

19. A lady made 56 patches for a quilt. Each patch was octagonal in shape, was made of 5 squares 3 inches in dimension when finished, and 4 triangular pieces, as shown in the figure. (Are the patches regular octagons?) The blocks were put together with plain muslin as shown in the figure. A strip of muslin of uniform width forms the border. The quilt when finished is $2\frac{1}{2}$ yards long and $2\frac{1}{4}$ yards wide. How much muslin did she use? Make no allowance for seams.



20. The length of the shorter side of a right triangle is $5\frac{1}{2}$ units more than $\frac{1}{2}$ of the other. The hypotenuse is 3 units less than twice the shorter side. What is the length of each side and of the hypotenuse?

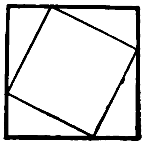
21. (*a*) Give three sets of numbers, other than the numbers 3, 4, 5, or multiples of them, which can be used as the sides of a right triangle.

21. (*b*) Show that $m^2 - n^2$, $2\,mn$, $m^2 + n^2$ are the sides and hypotenuse of a right triangle. Show this by making use of the Pythagorean theorem.

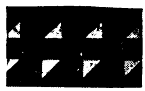
22. By substituting different numbers for m and n in Exercise 21, make a table of ten sets of numbers that may be used as the sides of a right triangle.

23. From the table of Exercise 22 write three problems concerning right triangles.

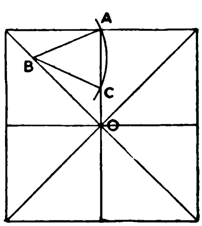
24. In the adjoining figure let a be the length of one side of the square. Calculate the length of the oblique lines in terms of a. Calculate the area of each part of the figure. Construct the figure letting $a = 2$ inches. Substitute 2 for a in your answer and compare results with those obtained by drawing.

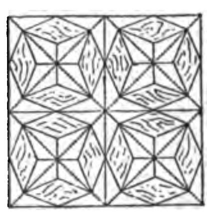


25. This is a design for a parquetry border. Construct this design, making your drawing 4 inches wide. Compute the area of each part.

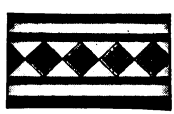


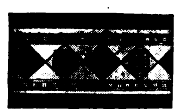
26. Construct this parquetry design for a border based on a regular octagon. (*Suggestion.* Lay off OB equal to OA. Then AB is the side of a regular octagon.) Prove this by proving the sides equal and the angles equal.



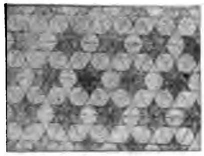


27. Construct parts of borders, four times as wide as in these illustrations.





28. Construct a parquetry design like the illustration so that the dark star shall have sides one-half inch long. Notice that the figure can be built up from a system of diagonal lines. Calculate the area of one of the stars in your drawing.

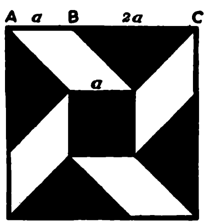


29. Construct this parquetry border so that the lines of your figure shall be four times as long as those of the illustration.

30. Construct part of this field so that the lines of your figure shall be four times as long as in the illustration. Note the parallel lines as a help to making the figure.

31. Construct part of this field using sets of parallel lines placed far enough apart to make the hexagon come out one inch on each side.

32. Construct the following figure, taking $a = 1$ inch. Find the area of each part.



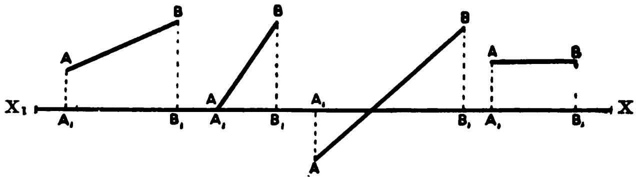
What is the perimeter of one of the black triangles, and of one of the parallelograms?

What are their areas?

117. Definition. The **projection** of a point on a line is the **foot of the perpendicular** drawn from the point to the line.

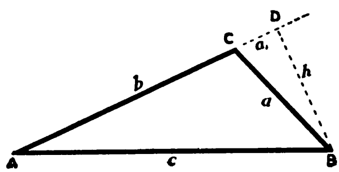
From what two Latin words is the word projection derived? Do you find any connection between the original meaning and the meaning here?

If we draw a line-segment AB and another unlimited line XX_1 in the same plane, and if we imagine every point on AB to be projected upon XX_1 then the segment of XX_1 which contains all of these projections is called the **projection of line-segment** AB upon line XX_1.



In each case A_1B_1 is the projection of AB upon XX_1.

118. Theorem X. *In an obtuse angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the other two sides, increased by twice the product of one of those sides by the projection of the other side on the line of that side.*



Given the triangle ABC with the obtuse angle ACB.

The lengths of sides BC, CA, and AB respectively are a, b, c; a_1 the projection of a upon the line of side b; h is the perpendicular used to project a upon line of b.

To prove that $c^2 = a^2 + b^2 + 2a_1b$.

Analysis. To prove this we find in our figure a right triangle of which c is the hypotenuse, and then by means of the

Pythagorean theorem we can give the value of c in terms of other lines. When we have done this, we shall find in the equations that we have h^2 and a_1^2 which are not called for in the proposition, while a^2 does not appear in the equation. If now it is possible to substitute a^2 for $h^2 + a_1^2$ in the equation, we shall have proved our theorem.

Proof. ABD is a right triangle. Why?

$\therefore$ $c^2 = h^2 + (a_1 + b)^2$. Why?

$= h^2 + a_1^2 + 2\,a_1b + b^2$. Why?

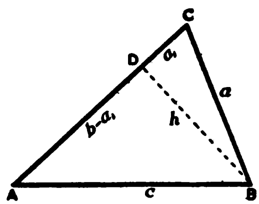
But $h^2 + a_1^2 = a^2$. Why?

Substituting a^2 for $h^2 + a_1^2$, we have

$$c^2 = a^2 + 2\,a_1b + b^2.$$

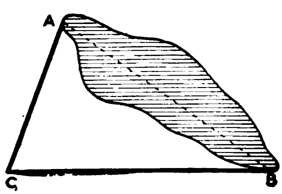
State the theorem proved.

119. Theorem XI. *In any triangle the square on the side opposite an acute angle is equal to the sum of the squares on the other two sides, diminished by twice the product of one of those sides and the projection of the other side upon it.*



Give the proof of this on the same plan as in the preceding theorem, noting that we now have $c^2 = h^2 + (b - a_1)^2$.

120. Exercises. If an engineer wishes to find the distance across a swamp which is impassable, he may use the following plan:

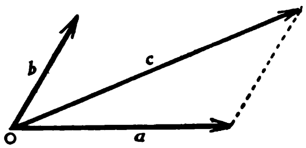


He places his transit at the point C from which he may see and go to both ends of the swamp. By sighting from C to A and from C to B, he can find angle BCA. (The teacher should explain the plan on which the transit works.) He can measure the distance from C to A and from C to B. From these data he can find the distance AB by making use of theorems IX, X, and XI, according to the size of $\angle BCA$.

From the following data compute this distance, taking $CB = a$ and $CA = b$. Draw the figures.

1. $a = 24$ yds., $b = 7$ yds., $\angle BCA = 90°$.
2. $a = 40$ yds., $b = 50$ yds., $\angle BCA = 150°$.
3. $a = 100$ yds., $b = 130$ yds., $\angle BCA = 30°$.
4. $a = 120$ yds., $b = 75$ yds., $\angle BCA = 60°$.
5. $a = 60$ yds., $b = 120$ yds., $\angle BCA = 120°$.

It is shown in physics that the resultant of two forces acting at the same instant upon an object is the same as though one of the forces acted and when this action finished, the second force acted. In the adjacent figure, if two mallets hit a ball at the same instant, the first of which would send the ball the distance a feet in the direction indicated, and the second the distance b in the direction indicated, the resultant distance would be the distance c in the direction indicated.



Also the velocity with which an object will move when acted upon by two forces can be obtained if we know the velocity due to each of the forces, and the angle at which they act with each other.

From drawings on cross section paper measure the resultant force or velocity in the following cases; $\angle ab$ means the angle between a and b.

6. $a = 25, b = 50, \angle ab = 54°$.
7. $a = 70, b = 45, \angle ab = 170°$.
8. $a = 100, b = 120, \angle ab = 93°$.

In the following, after drawing, compute the resultant force or velocity and compare answers.

9. $a = 100, b = 20, \angle ab = 30°$.

10. $a = 10, b = 50, \angle ab = 90°$.

11. $a = 100, b = 20, \angle ab = 90°$.

12. $a = 100, b = 20, \angle ab = 150°$.

13. $a = 40, b = 40, \angle ab = 120°$.

(For illustration of solution of following exercises, see First Course, pages 194, 195.)

14. Two forces are acting at right angles to one another. The smaller force is 5 pounds less than $\frac{1}{2}$ the larger force, and the resultant force is 9 pounds more than the larger. What is the number of pounds in the resultant force? Draw.

15. If rain-drops which are falling vertically at the rate of 120 feet per second enter an air current which is moving horizontally at 50 feet per second, what will be the resultant velocity of the rain-drops? Make a drawing showing their direction.

16. From a moving body a particle is thrown at right angles to the direction in which the body is moving. The velocity of the particle as it leaves the body is 4 miles per second more than the velocity of the body, while the velocity at which it is thrown from the body is $\frac{2}{3}$ of a mile more per second than $\frac{1}{3}$ of the velocity with which it leaves the body. What is the velocity of the particle? Draw.

17. A ship is sailing due southeast at the rate of $11\frac{1}{8}$ miles an hour in an ocean current which is flowing due north-east at the rate of $2\frac{1}{2}$ miles an hour. What is the resultant speed of the ship? Show by a diagram the direction in which the ship actually moves.

18. How much sod will it take to resod a corner of a lawn, which is in the shape of a right-angled triangle, if the length of the longer side is 3 feet more than that of the shorter side, and the length of the shorter side is 4 feet more than $\frac{1}{3}$ of the length of the hypotenuse?

121. Summary of Chapter III.

Equal or Equivalent Figures.

Definition of Equal Figures, Addition and Subtraction of Areas.

Definition of Base and Altitude of Parallelogram, Triangle, Trapezoid.

Theorem I. Parallelograms on the same or equal bases and between the same parallel lines are equal.

Corollary 1. Parallelograms having equal bases and altitudes are equal.

Corollary 2. A parallelogram is equal to a rectangle which has an equal base and stands between the same parallel lines.

Theorem II. Triangles which stand on the same or equal bases and between the same parallel lines are equal.

Corollary 1. Triangles having equal bases and equal altitudes are equal.

Corollary 2. A triangle equals half a parallelogram on the same or equal bases and between the same parallel lines.

Theorem III. A trapezoid is equal to half a rectangle whose base is the sum of the two parallel sides, and whose altitude is the altitude of the trapezoid.

Assumption. The area of a rectangle equals the product of its base by its altitude.

Corollary 1. The area of a parallelogram equals the product of its base by its altitude.

Corollary 2. The area of a triangle equals the product of half its base by its altitude.

Corollary 3. The area of a square is equal to the square of one of its sides.

Corollary 4. The area of a trapezoid equals the product of half the sum of its bases by its altitude.

Sums and Differences of Areas.

Theorem IV. The rectangle of two given lines equals the sum of the rectangles contained by one of them and the several segments into which the other is divided.

Theorem V. The square on the sum of two lines equals the sum of the squares on those lines plus twice their rectangle.

Theorem VI. The rectangle of a given line and the difference of two other lines equals the difference of the rectangles of the first line and each of the others.

Theorem VII. The square on the difference of two lines is equal to the sum of the squares on those lines minus twice their rectangle.

Theorem VIII. The difference of the squares on two lines is equal to the rectangle of the sum and difference of those lines.

Theorem IX. The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.

Problem I. To construct a right triangle when two sides are given.

Problem II. To construct a right triangle when a side and hypotenuse are given.

Problem III. To construct a square that will be the sum of two given squares.

Problem IV. To construct the square root of a number.

Definition. Projections.

Theorem X. In an obtuse angled triangle the square on the side opposite the obtuse angle is equal to the sum of the squares on the other two sides, plus twice the product of one of these sides and the projection of the other side on the line of that side.

Theorem XI. In any triangle the square on the side opposite an acute angle is equal to the sum of the squares on the other two sides, minus twice the product of one of these sides by the projection of the other side upon it.

1. Of what subject matter does Chapter III treat?

2. Can figures be equal without being congruent? Explain.

3. Name various figures that you know to be equal, as brought out in this chapter. Name figures that you know to have half the area of other figures.

4. To what subject in algebra is part of this chapter closely related?

5. What famous theorem is given in this chapter? By what name is it known and why?

6. Go through the list of theorems (page reference). Do you find theorems that have been proved without the use of theorems of Chapter II?

The theorems of this chapter are selected from Book II of Euclid's Geometry.

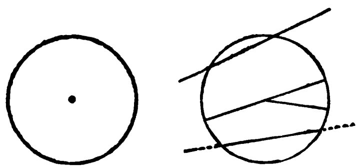
CHAPTER IV

PART I—CIRCLES. CENTRAL ANGLES. CHORDS.
PART II—INSCRIBED ANGLES.

PART I—CIRCLES. CENTRAL ANGLES. CHORDS.

122. Definitions. See also Articles **41** and **42.**

A **circle** is the portion of a plane bounded by a curved line, all points of which are equidistant from a point within called the center.



The curved line is called the **circumference** of the circle.

Note. **This is true in the elementary geometry, but as you go farther in your work, you will find often that the curved line is called the circle.**

A straight line terminated by the center and the circumference is called a **radius** of the circle.

A straight line passing through the center of the circle and terminated by the circumference is called a **diameter** of the circle.

A line cutting the circumference is called a **secant.**

The segment of a secant cut off by a circumference is called a **chord.**

Equal circles are those the **diameters** of which are **equal,** or the **radii** of which are **equal.**

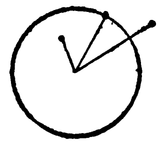
Exercises. 1. A circle passes through the four corners of a square. Show that a diagonal of the square is a diameter of the circle. What is the diameter of the circle if the side of the square is a units?

2. Draw a circle on a given line-segment as diameter.

123. Postulates for the Circle. The following assumptions are made:

Post. VII. *A circumference may be drawn with any point as center and with any given line-segment as radius.*

Post. VIII. *A point is within, on, or without a circumference according as its distance from the center is less than, equal to, or greater than the radius.*

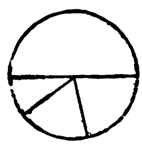


Exercise. Draw a square with sides 2 inches long, and draw circles on each side of the square as diameters. Show that these circles are equal and that they will all pass through the center of the square.

124. Definitions. An **arc** is any part of a circumference.

One-half of a circumference is called a **semicircumference.**

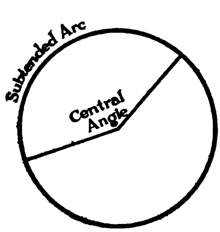
A **sector** is the portion of a plane bounded by two radii of a circle, and the arc which they intercept.

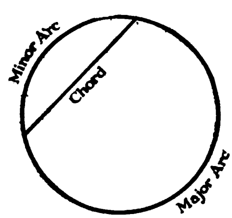


One-half of a circle is called a **semicircle.**

A **central** angle is an angle whose arms are radii.

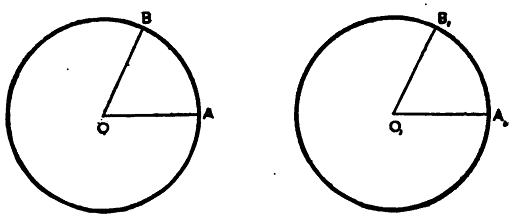
We use the expression: A central angle stands on an arc, or **intercepts** the arc. The arc **subtends** the central angle.





A chord subtends two arcs. The smaller arc is called the **minor arc,** and the larger arc is called the **major arc.**

125. Theorem I. *In the same or equal circles, if two central angles are equal, their intercepted arcs are equal.*



Given the equal circles whose centers are O and O_1 with the equal central angles AOB and $A_1O_1B_1$ intercepting the arcs AB and A_1B_1 respectively.

To prove that arc AB is equal to the arc A_1B_1.

Analysis. Since this is the first proposition that we have to prove with reference to circles, we shall use the method of superposition. Placing the circle whose center is O on the circle whose center is O_1, we must show that every point in arc AB falls upon a corresponding point in arc A_1B_1.

Proof. Place the circle whose center is O on the circle whose center is O_1, so that the $\angle AOB$ coincides with $\angle A_1O_1B_1$. Then A falls on A_1. Why?

Also B falls on B_1. Why?

Then every point in arc AB falls upon a corresponding point in arc A_1B_1. Why?

State proposition proved. State the converse theorem.

Corollary 1. *A diameter divides a circumference into two equal parts.* Explain.

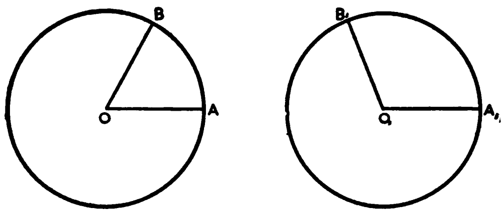
Corollary 2. *In the same or equal circles sectors which have equal angles are equal.* Why?

Corollary 3. *A diameter divides the circle into equal parts.* Why?

126. Theorem II. State the converse of Theorem I. (See Summary.) Prove by placing the two circles together, so that the centers fall together, and the given equal arcs coincide. Show that the angles coincide and are therefore equal.

Corollary 1. State and prove converse of Theorem I, Corollary 2.

127. Theorem III. *In the same or equal circles if two central angles are unequal, the arcs which they intercept are unequal, and the greater angle intercepts the greater arc.*



Given the equal circles whose centers are O and O_1 with the unequal central angles AOB and $A_1O_1B_1$ intercepting the arcs AB and A_1B_1 respectively, angle AOB being less than angle $A_1O_1B_1$.

To prove that arc AB is less than arc A_1B_1.

Analysis. We shall prove this by placing the circle whose center is O on the circle whose center is O_1, with center O on O_1, and the arm OA on arm O_1A_1. From this we can show that the initial ends of the arcs AB and A_1B_1 coincide, and that the final ends do not coincide.

Proof. Placing the circle with center O on the circle with center O_1, O being placed on O_1, and OA on O_1A_1, point A will fall on point A_1. Why?

OB will fall between O_1A_1 and O_1B_1, because $\angle AOB < \angle A_1O_1B_1$.

$\therefore$ B will fall between A_1 and B_1.

$\therefore$ arc $AB <$ arc A_1B_1.

State the theorem proved. State the converse theorem.

Theorem IV. State and prove the converse of Theorem III. (See Summary.)

Theorems I and III are usually stated together as one theorem.

128. Measurement of Arcs. Since equal angles intercept equal arcs, each of the 360 equal angles into which we divided a perigon in order to establish the degree unit, will intercept one three hundred and sixtieth of a circumference described about the vertex as a center. From this fact we speak of one three hundred and sixtieth of a circumference as **an arc of one degree.** Thus a unit for measuring an arc is established. An arc of 20 degrees is intercepted by a central angle of 20 degrees.

You will frequently see the expression that "an arc measures an angle." We shall use this expression in our work and it is to be understood that the unit for measurement of arc is derived as above from the unit of angle.

129. Theorems I, III. *In the same circle or equal circles if two central angles are equal, the intercepted arcs are equal, and of two unequal central angles the greater angle intercepts the greater arc.*

This theorem suggests the dependence of one magnitude on another, in this case the dependence of arc on central angle, so we will now consider this subject. At this time it will be well for the student to review First Course, pages 206 to 214.

130. Definition. One quantity is said to be a **function** of another quantity when it depends on that quantity for its value.

From the above theorem we may state that *the length of the arc depends on the size of the central angle*, that is,

The length of the arc is a **function** *of the size of the central angle, if the circle remains the same size.*

Other Examples:

In general, the size of a tree depends on the number of years that it has been growing, that is,

The size of a tree is a function of the number of years of growth.

The rise and fall of the mercury in the thermometer depends on the temperature, that is,

The height of the mercury in a thermometer is a function of the temperature.

The weight of a ball of given material depends on the volume, that is,

The weight of a ball is a function of the volume. Let the student give other illustrations of this kind.

Drawing illustrations from our past work in geometry, we have:

The area of a rectangle is a function of the altitude if the base remains the same, or

The area of a rectangle is a function of the base if the altitude remains the same.

Let the student go through the theorems which have been studied and decide upon those that involve the dependency of one magnitude upon another, and re-state some of them, using the word function.

131. Algebraic Expression of the Function Idea.

In the illustration of the rectangle

$$a = bh$$

is the algebraic expression for the area of a rectangle.

Then $\quad a = f(h),\quad$ (read "*a is a function of h*")

where $\quad f(h) = bh,$

is the algebraic way of stating that if the base remains the same, the area of a rectangle is a function of its altitude.

Also $\quad a = f(b),\quad$ (read "*a is a function of b*")

where $\quad f(b) = bh,$

is the algebraic way of stating that, if the altitude remains the same, the area of a rectangle depends on its base.

Finally $a = f(b, h)$ (read "*a is a function of b and h*")
is the algebraic way of stating that if the base and altitude both change, the area of the rectangle depends on them both.

132. Exercises.

1. Make drawings illustrating each of the above statements. (Cross-section paper should be used.)

2. As shown in the illustration above, give algebraic statements for the area of a *parallelogram.*

3. Make similar statements for the changes in the area of a *triangle.*

4. If r, x, y are the hypotenuse and the sides of a right triangle, we have $r^2 = x^2 + y^2$.

Tell what the following expressions mean:

$$r = f(x),$$
$$r = f(y),$$
$$r = f(x, y).$$

It will be noticed in the above exercises there are two kinds of quantities, namely, variables and constants. Of the variable numbers there are two kinds, the independent variables or those which we change at will, and the dependent variables or those which change because another number changes. The dependent variables are functions of the independent variables.

In the theorems studied in the future note carefully these quantities.

Theorems II and IV may be stated in one theorem.

133. Theorems II, IV. *In the same circle or equal circles, if arcs are equal, they are intercepted by equal central angles, and of two unequal arcs the greater arc is intercepted by the central greater angle.*

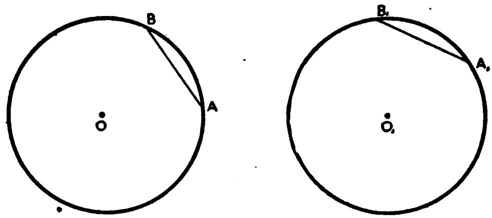
In this theorem, what is the independent variable? What is the dependent variable? What is the constant?

Use the word function to state the relation between the independent and dependent variables.

Explain how the independent variable of Theorem III becomes the dependent variable in Theorem IV, and vice versa.

THEOREMS ON ARCS AND CHORDS

134. Theorem V. *In the same or equal circles, if two arcs are equal they are subtended by equal chords.*



Given the equal circles whose centers are O and O_1 with the arcs AB and A_1B_1 equal, subtended by the chords AB and A_1B_1 respectively.

To prove that chord AB is equal to chord A_1B_1.

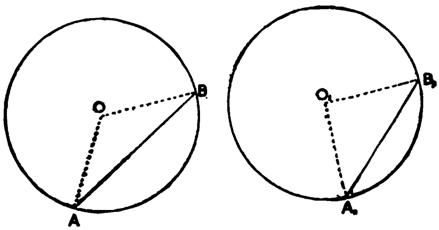
Analysis. We can prove chord AB and chord A_1B_1 equal if we draw the radii OA, OB, O_1A_1, O_1B_1, and prove the triangles thus formed congruent. We can prove the triangles congruent, if we can prove the $\angle s$ AOB, $A_1O_1B_1$ equal, since we know the radii are equal. $\angle AOB$ equals $\angle A_1O_1B_1$ if arcs AB and A_1B_1 are equal. But these are equal by hypothesis.

Proof. Make this proof complete.

135. Theorem VI. State the converse of Theorem V. (See Summary.)

Prove by using congruent triangles.

136. Theorem VII. *In the same or equal circles, if two minor arcs are unequal, the chords by which they are subtended are unequal, and the greater arc is subtended by the greater chord.*



Given the equal circles whose centers are O and O_1, with minor arc AB greater than minor arc A_1B_1.

To prove that chord AB is greater than A_1B_1.

Analysis. To prove this we form two triangles OAB and $O_1A_1B_1$ by drawing radii OA, OB, O_1A_1, O_1B_1. We can prove line AB greater than A_1B_1 if we can show that the sides OA and OB are equal respectively to O_1A_1 and O_1B_1 and that the included angle AOB is greater than the included angle $A_1O_1B_1$.

Proof.	arc $AB >$ arc A_1B_1.	Why?
	$\angle AOB > \angle A_1O_1B_1$.	Art. **133.**
	$OA = O_1A_1$,	
	$OB = O_1B_1$.	Why?
$\therefore$	side $AB >$ side A_1B_1.	Art. **88.**

State theorem proved.

State converse of this theorem.

Exercise. A circumference is divided into four equal arcs, and their chords are drawn.

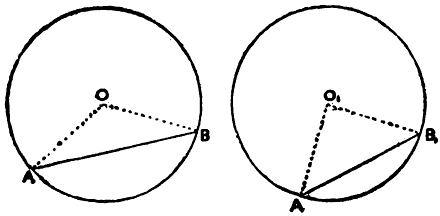
(a) Show that the sides of the quadrilateral are equal.

(b) Show that the diagonals of the quadrilateral are diameters.

(c) Show that the quadrilateral is a parallelogram.

(d) Show that the quadrilateral is a rectangle.

137. Theorem VIII. *In the same or equal circles if two chords are unequal, they subtend unequal minor arcs, and the greater chord subtends the greater arc.*



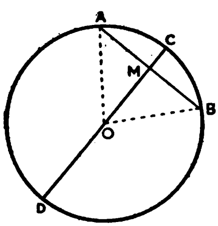
Given the equal circles whose centers are O and O_1, with chord AB greater than A_1B_1.

To prove that arc AB is greater than arc A_1B_1.

Analysis. To prove this we form two triangles OAB and $O_1A_1B_1$ by drawing radii OA, OB, O_1A_1 and O_1B_1. To prove arc AB greater than arc A_1B_1, we shall prove $\angle AOB$ greater than $\angle A_1O_1B_1$. We can do this by applying the theorem of Art. **89**, page 75, to the two triangles.

Proof.	side $AB > A_1B_1$.	Why?
	side $OA = O_1A_1$,	
	side $OB = O_1B_1$.	Why?
$\therefore$	$\angle AOB > \angle A_1O_1B_1$.	Art. **89.**
$\therefore$	arc $AB >$ arc A_1B_1.	Art. **128.**

138. Theorem IX. *A diameter which is perpendicular to a chord bisects the chord and its subtended arc.*

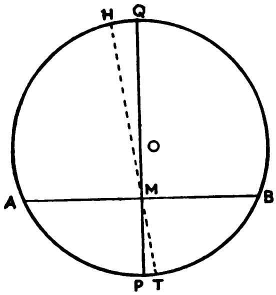


Prove by drawing in radii and showing congruent triangles.

139. Theorem X. *If a diameter bisects a chord, it is perpendicular to the chord.*

Prove by drawing radii and proving congruent triangles.

140. Theorem XI. *The perpendicular bisector of a chord passes through the center of the circle, and bisects the subtended arc.*



Given the circle with center O and chord AB. Line PQ is perpendicular to AB at the mid-point M.

To prove that PQ passes through center O.

Analysis. We shall suppose PQ does not pass through center O. We shall draw the diameter HT through the mid-point M. We can prove that PQ passes through O if we can show that it coincides with HT. We can show that it coincides with HT if we can show that it makes the same angle with AB that HT does. We can show that both PQ and HT make right angles with AB.

Proof.	$HT \perp AB$ at M.	Why?
$\therefore$	$\angle BMH$ is a right angle.	Why?
	$PQ \perp AB$ at M. Why?	(Hypothesis.)
$\therefore$	$\angle BMP$ is a right angle.	
$\therefore$	PQ and HT coincide.	Why?
$\therefore$	PQ passes through center O.	Why?

State proposition proved.

Corollary. *In general, two chords can not bisect each other.*

What is the exception?

Exercises.

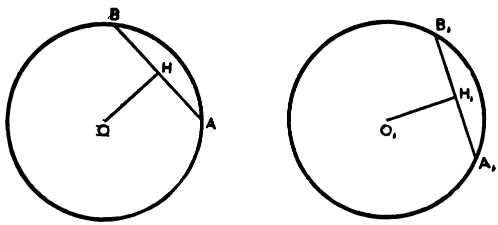
1. Through a given point in a given circle, draw a chord which shall be bisected at the point.

2. The line joining the mid-points of two chords passes through the center. Show that the chords are parallel.

THEOREMS ON CHORDS AND DISTANCES FROM THE CENTER

141. Theorem XII. *In the same circle or equal circles equal chords are equidistant from the center.*

Note. By distance from the center we mean the perpendicular distance.



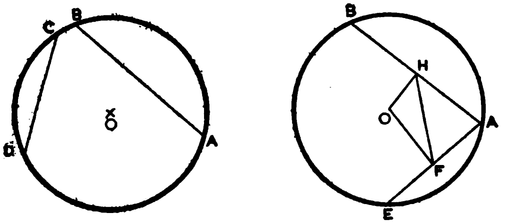
Given the equal circles whose centers are O and O_1 with the equal chords AB and A_1B_1 at the distances OH and O_1H_1 respectively.

To prove OH equal to O_1H_1.

Prove by drawing radii OA and O_1A_1 and showing congruent triangles.

142. Theorem XIII. State and prove the converse of Theorem XII. (See Summary.)

143. Theorem XIV. *In the same or equal circles, if two chords are unequal, they are unequally distant from the center and the greater chord is the less distant.*



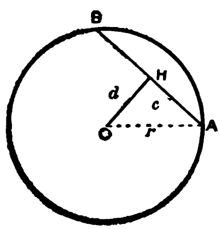
Given the circle whose center is O, with chord AB greater than chord CD.

To prove that AB lies nearer O than does CD.

Analysis. Suppose chord AE drawn equal to chord CD. Let OH and OF be ⊥s from O to AB and AE respectively. Then OH and OF are sides of a triangle, and we can prove $OH < OF$ if we can prove $\angle OFH < \angle OHF$. (See Art. **70.**) But these angles are complements of $\angle AFH$ and $\angle AHF$ respectively in $\triangle AHF$. From what you know about the lengths of AF and AH, can you prove $\angle AFH > \angle AHF$? (See Art. **53.**) Then what follows regarding $\angle OFH$ and $\angle OHF$? Then what do you conclude about OH as compared with OF.

Proof. Let the student give the proof in full.

Algebraic Proof. Given the circle whose center is O with the chord AB at the distance OH from the center. (Figure below.)



To prove that if chord AB is to become longer, it must move nearer to the center.

Let d = the number of units in OH;
c = the number of units in AH;
r = the number of units in OA.
$\triangle AHO$ is a right triangle. Why?
Therefore $d = \sqrt{r^2 - c^2}$. Why?

We may regard c as an independent variable in this expression. Then d will become the dependent variable. In other words we shall regard $d = f(c)$.

The quantity $r^2 - c^2$ will become smaller as the value of c becomes larger, because the less remainder is obtained by subtracting the greater magnitude. Therefore d becomes smaller as c becomes larger.

State the theorem proved.

144. Theorem XV. State the converse of Theorem XIV. (See Summary.)

Give proofs similar to those of Theorem XIV.

Corollary. *The diameter is the longest chord in a circle.*

145. Exercises. Geometric and Algebraic.

1. If through a point on the circumference of a circle, chords are drawn which make equal angles with the radius drawn to that point, the chord will be equal.

Suggestion. Prove that they are equally distant from the center.

2. A chord is drawn in a circle; another chord is drawn through its mid-point; another chord is drawn through the mid-point of the last, and so on. Prove that these chords continue to grow longer. Point out the independent and dependent variables, and the constant.

Suggestion. Prove that the distance from the center is less for each successive chord.

3. A chord MN is drawn in a circle. Chords XY are drawn so that they continually have their mid-points on the chord MN. What is the length of the longest and of the shortest chord XY.

Point out the independent, the dependent variables and the constant.

4. A circle whose center is on the bisector of an angle cuts off equal chords, if any, from the arms of the angle.

5. If two equal chords intersect within a circle, one set of lines joining their ends are equal.

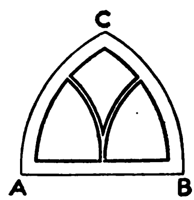
6. Of all the chords that can be drawn through a point within a circle the one perpendicular to the diameter through that point is the shortest.

7. If from the ends of the diameter of a circle lines are drawn perpendicular to a secant of the circle, prove that the segments cut between the circumference and the feet of the perpendiculars are equal.

Make two figures, one in which the secant cuts the diameter, and the other in which it does not cut the diameter.

Suggestion. Draw a line from the center of the circle perpendicular to the secant.

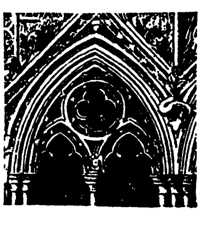
8. Figure ABC is a Gothic arch. Arcs AC and BC are drawn from A and B as centers and with radius equal to line AB. Construct such an arch.



Show that each arc is one-sixth of a circumference.

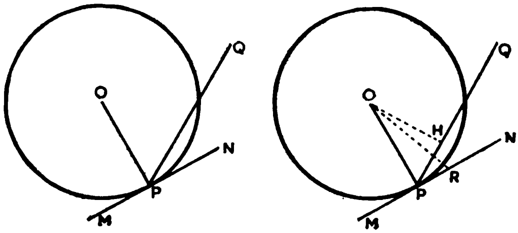
Find the height of the arch if line AB = 60 inches.

If the height of the arch is to be 30 inches, how wide must it be?



9. Construct a figure in which two smaller and equal Gothic arches are contained in a larger arch. Find the height of each arch if the large arch is 4 inches wide.

146. Theorem XVI. *Of all lines passing through a point on a circumference, the perpendicular to the radius drawn to that point is the only one that does not meet the circumference again.*



Given the circle whose center is O, with point P on the circumference.

Through P line MN is drawn perpendicular to the radius OP. PQ is drawn oblique to OP.

To prove that MN does not meet the circumference again and that PQ does meet it again.

Analysis. In order to prove that MN does not meet the circumference again, we prove that every point on MN other than P is more distant from O than is P. In order to prove that PQ cuts the circumference again, we must prove that there exists at least one point on PQ at a less distance than OP from O.

Proof. Select any point other than P on MN, say point R. Join O and R.

$$OR > OP. \quad \text{Why?}$$

Therefore MN does not meet the circumference again.

Since OP is not perpendicular to PQ, drop a perpendicular to PQ from O. Call it OH.

$$OH < OP. \quad \text{Why?}$$

Therefore there is a point H on PQ that is nearer O than is P, and hence lies within the circle. Art. 123. Hence the PQ cuts the circumference again. Why?

Exercises.

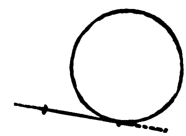
1. In the figure on page 127, suppose a point T marked on PN, so that $PT = OP$. At T draw a line perpendicular to PN. Show that it will meet the circle in only one point.

2. Show how to draw a circle with a given radius and which shall be met by a given line in only one point. Draw several such circles.

3. Draw a circle with a radius of 1.5 inches, and let P be a point 2.5 inches from the center. With P as center and a radius 2 inches draw an arc cutting the circle in Q. Will line PQ cut the circle once, twice, or not at all? Prove.

147. Definition.

A **tangent** to a circle is an unlimited straight line which meets the circumference in only one point. This point is called the **point of tangency**, or the **point of contact.**



When we speak of a **tangent from a point to a circle,** we mean the segment of the tangent between the point and the point of contact.

148. Theorem XVII. *One and only one tangent can be drawn to a circle at a point on its circumference.*

Suggestion. Draw the radius to the point and try to draw two tangents at the point. Explain why this can not be done.

149. Theorem XVIII. *A tangent is perpendicular to the radius drawn to the point of tangency.*

150. Theorem XIX. *A line perpendicular to a radius at its extremity on the circumference is tangent to the circle.*

151. Theorem XX. *The center of a circle lies on a perpendicular to a tangent at the point of tangency.*

Suggestion. Suppose that the center does not lie on the perpendicular. From the center draw a radius to the point of tangency. Prove that this radius coincides with the perpendicular, and hence that the center lies on the perpendicular.

152. Theorem XXI. *The perpendicular from the center of a circle to a tangent meets it at the point of tangency.*

Suggestion. Suppose that the perpendicular does not go through the point of tangency. Draw a radius to the point of tangency and prove that it coincides with the perpendicular, and hence that the perpendicular goes through the point of tangency.

153. Exercises.

1. If the distance from a point to the center of a circle equals the diameter, determine the size of the angle between the tangents drawn from the point to the circle. Prove. Apply Art. **72**, Ex. 5.

2. If two tangents are drawn from a point to a circumference of a circle the angle contained by them is supplementary to the angle contained by the radii drawn to the point of tangency.

3. If two diameters are drawn at right angles to one another, and tangents are drawn through their ends, the figure formed by the tangents is a square.

4. If three diameters of a circle are drawn so that they divide the perigon about the center into six equal parts, and if tangents are drawn through their ends, a regular hexagon will be formed.

5. If in Exercise 4 you increase the number of diameters and tangents, variable quantities will appear.

What is the independent variable?

What are the dependent variables?

What is the constant?

Give answer using the word function.

Examine the variables and tell whether the dependent variables increase or decrease as you change your independent variable.

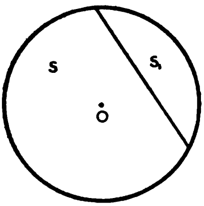
If they increase tell how large they may become. If they decrease tell how small they may become.

6. If two tangents to a circle are parallel, the line joining their points of contact is a diameter.

7. A circle tangent to the three sides of a square must be tangent to the fourth side.

8. Two circles of radii 1 and 2 inches respectively have the same center. A chord of the larger circle is tangent to the smaller. How long is the chord?

154. Definition. A **segment of a circle** is either of the two portions into which a chord divides the circle.



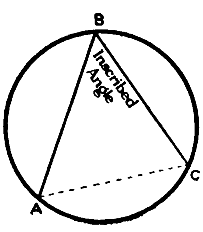
The portion S_1 is called the **minor segment.**

The portion S is called the **major segment.**

If two segments are equal they are **semicircles.**

An angle formed by two chords meeting on a circumference is called an **inscribed angle.**

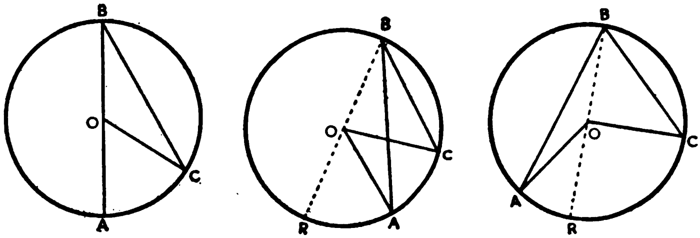
The arc which the arms of an inscribed angle cut off is said to **subtend** the angle, and the angle is said to stand on or **intercept** that arc.



In the figure, angle ABC is said to stand on, or intercept, arc AC and arc AC is said to subtend the inscribed angle ABC.

If a chord is drawn from A to C, we say that angle ABC is inscribed in the major segment AC and intercepts the minor arc AC.

155. Theorem XXII. *An inscribed angle equals half the central angle standing on the same arc.*



Given the angle ABC inscribed in the circle whose center is O, and the central angle AOC, standing on the same arc AC.

To prove that angle ABC is equal to half angle AOC.

Analysis. There are three possibilities. One arm of the angle may be a diameter, or both arms may lie on one side of the diameter, or one arm may be on one side of the diameter and the other on the other side, as shown in the three figures above.

In the first figure we can prove that $\angle AOC$ is twice $\angle ABC$, if we can prove that it is equal to the sum of $\angle$s ABC and BCO, and can then show that $\angle$s ABC and BCO are equal. We can show that $\angle$s ABC and BCO are equal if we can show that $\triangle OCB$ is isosceles.

Proof. (First figure.) $\triangle OCB$ is isosceles. Why?

$\therefore$ $\angle ABC = \angle BCO$. Why?

$\angle AOC$ is an exterior angle to $\triangle OCB$.

$\therefore$ $\angle AOC = \angle OBC + \angle BCO$. Why?

$\therefore$ $\angle AOC = 2 \angle ABC$. Why?

$\therefore$ $\angle ABC = \frac{1}{2} \angle AOC$.

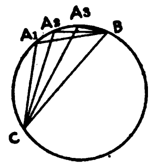
State the theorem.

In the second figure, to prove $\angle AOC = 2 \angle ABC$, we make application of the truth proved in the first figure. Draw in diameter BR, and show that $\angle ABC$ is the result of subtracting $\angle RBA$ from $\angle RBC$, and $\angle AOC$ is the result of subtracting $\angle ROA$ from $\angle ROC$.

Give this proof.

In the third figure, after drawing in the diameter, show that $\angle ABC = \angle ABR + \angle RBC$, and $\angle AOC = \angle AOR + \angle ROC$.

Give this proof.

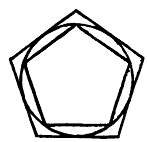


Corollary. *Angles inscribed in the same segment or in equal segments are equal.* Give reason.

Exercise. If chord BC is taken shorter and shorter, what change will take place in the angle inscribed in the minor segment? In the angle inscribed in the major segment?

156. Theorem XXIII. *An angle inscribed in a segment is greater than, equal to, or less than a right angle, according as the segment is less than, equal to, or greater than a semicircle.* Supply proof.

157. Definition. **A polygon** is said to be **inscribed** in a circle when its sides are chords of the circle. We also say that the circle is circumscribed about the polygon.



A **polygon** is said to be **circumscribed** about a circle, when its sides are tangent to the circle. We also say that the circle is inscribed in the polygon.

158. Exercises.

1. If an inscribed angle is 20 degrees, how many degrees in the intercepted arc? If it is 50 degrees? If 110 degrees? If 178 degrees?

2. Angle ABC is inscribed in a circle. If arc AC is the length of a radius, what is the value of angle ABC?

3. What arc is intercepted by an inscribed angle which equals a radian?

4. Draw an inscribed angle which will equal $\frac{1}{3}\pi$ radians.

5. In the figure in Exercise 2 let the points B and A remain fixed and let the point C move around the circumference from A to B. State the least value of $\angle ABC$. How large may it become?

6. If points A and C remain fixed and point B moves from A to C tell all the values that angle ABC may have.

7. If two equal arcs are laid off on a circumference, and their ends are joined, there will be two chords which intersect and two chords which are parallel. Prove that the chords that intersect are equal and divide each other into mutually equal parts, and also prove that the other two chords are parallel.

8. A triangle, one of whose angles is 25 degrees, and another angle 100 degrees, is inscribed in a circle. Tell, with reason, the number of degrees in the arcs subtended by the sides of the triangle.

9. With the extremities of any diameter as centers and with a radius equal to the radius of the circle arcs are struck cutting the circumference. Show that the points so obtained together with the extremities of the diameter form the vertices of a regular inscribed hexagon, by showing that the angles are equal and the sides are equal.

Calculate the perpendicular distance from the center of the circle to one of the sides, if radius of the circle is r.

10. Tell the number of degrees in each of the arcs subtended by the side of a regular hexagon inscribed in a circle; of a nonagon; of a pentagon.

11. Prove that the sum of the angles of a triangle is equal to a straight angle by inscribing it in a circle.

12. Prove similarly that the opposite angles of an inscribed quadrilateral are supplementary.

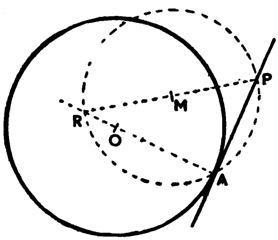
13. If a circle is drawn with one side of an isosceles triangle as a diameter, it will bisect the base.

14. Two chords perpendicular to a third chord at its extremities are equal.

15. If any number of angles are inscribed in the same segment of a circle, their bisectors meet in a point on the subtending arc.

We may use the fact that *an angle inscribed in a semicircle is a right angle,* to construct a tangent to a circle at a given point on its circumference.

159. Problem I. *To construct a tangent to a circle at a given point on its circumference.*



Given the circle whose center is *O*, with the given point *A* on the circumference.

To construct a tangent at point A.

Construction. Draw OA. With any center M (not on OA) and a radius MA draw a circle. Call the point where it cuts OA, R. Draw diameter RM, cutting the circle again at P. AP is tangent at A.

Proof. Give the proof.

Also solve this problem by using Art. **47,** Cor. 1.

Exercises.

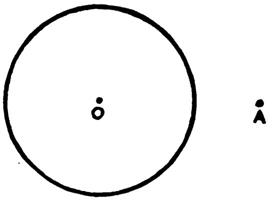
1. Show how to circumscribe an equilateral triangle about a given circle.

2. Show how to circumscribe a square about a circle

3. Divide a circumference into 6 equal parts by drawing chords equal to the radius.

Draw tangents at the points of division. What kind of a polygon is formed by these tangents?

160. Problem II. *To draw a tangent to a circle from an external point.*



Given the circle whose center is O, and the external point A.

To draw a tangent to the circle passing through point A.

Analysis. Since the tangent from A must meet the radius from O at right angles, and since to get this right angle, we may construct a semicircle which will contain it, namely a semicircle with OA as a diameter, bisect OA and with the mid-point as a center and one half OA as a radius draw a circle cutting the given circle in T and T_1. Draw AT and AT_1.

Proof. Give the proof.

Exercises.

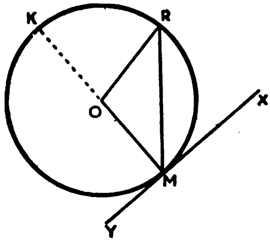
1. Show that the tangents from an exterior point to a circle are equal.

2. Draw a circle with radius 2 inches long. Mark an exterior point 1 inch from the circumference. Construct the tangents from this point to the circle. Calculate the lengths of these tangents. Check by measurement of drawing.

3. If the radius of a circle is 3 cm. long, how far from the center must a point be taken to make the tangents from the point 12 cm. long?

4. If two exterior points are unequally distant from the center, the tangents from the more remote point are the longer.

161. Theorem XXIV. *An angle formed by a tangent and a chord is equal to one-half the central angle standing on the same arc.*



Given the circle with center O, with the angle XMR formed by the tangent YX and the chord MR meeting at the point M. Also given the central angle MOR standing on the intercepted arc MR.

To prove that angle XMR is equal to one half of angle MOR.

Analysis. In order to prove that angle XMR is one-half of angle MOR, extend the radius MO through O making diameter MK.

Since $\angle XMR$ is the difference between right angle XMK and inscribed angle RMK, we have but to show that $\angle MOR$ is equal to the difference between a straight angle of which right angle XMK must be one-half, and $\angle ROK$ of which inscribed angle RMK is one-half.

Proof.

Str. $\angle MOK - \angle ROK = \angle MOR$. Why?

$\angle XMK = \frac{1}{2} \angle MOK$. Why?

$\angle RMK = \frac{1}{2} \angle ROK$. Why?

$\therefore \angle XMK - \angle RMK = \frac{1}{2}(\angle MOK - \angle ROK)$. Why?

$\therefore \angle XMR = \frac{1}{2} \angle MOR$. Why?

State the proposition proved.

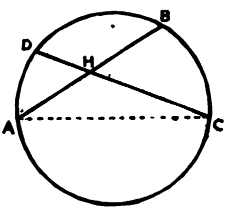
Corollary. *Tangents to a circle from an exterior point are equal.* Why?

Exercises.

1. In the figure of Theorem XXIV, $\angle RMY$ is also formed by a tangent and a chord. Prove that $\angle RMY$ equals half of the reflex angle ROM, which subtends the major arc RM.

2. In the figure of Theorem XXIV suppose R to move around on the circumference. What angles will change value?

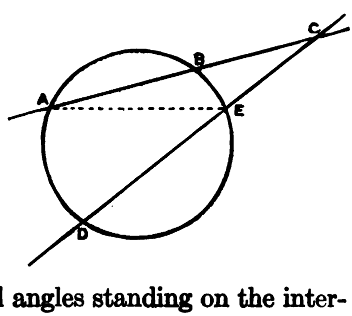
162. Theorem XXV. *An angle formed by two chords intersecting within a circle, is equal to one-half of the sum of the central angles subtended by the intercepted arcs.*



Suggestion. Draw a chord from one end of one of the chords to one end of the other. Show that the angle formed by the chords is equal to the sum of the two inscribed angles standing on the intercepted arcs (Art. 65, Cor. 1) and hence equal to half the sum of the central angles on these arcs.

163. Theorem XXVI. *An angle formed by two secants intersecting without the circle, is equal to one-half the difference between the central angles standing on the intercepted arcs.*

Suggestion. Draw a chord from the point where one of the secants cuts the circumference to the point where the other secant cuts the circumference.



Show that the angle formed by the secants is equal to the difference between the two inscribed angles standing on the intercepted arcs, Art. 65 Cor. 1, and hence equal to half the difference between the central angles standing on the intercepted arcs.

164. Theorem XXVII. *An angle formed by a secant and a tangent is equal to one-half the difference between the central angles standing on the intercepted arcs.*

Suggestion. Show that by turning one of the secants in the figure in Theorem XXVI, about the point of intersection, until points C and B come together, that theorem still holds for every position of the secant until points C and B coincide, and hence holds when they coincide.

This suggests that, by regarding the tangent as a special position of a secant, we may say that a tangent cuts a circle in two coincident points.

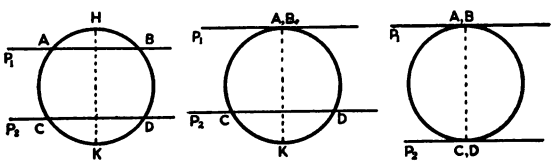
Exercises.

1. The angle between two chords AB and CD is 40°; arc AD is 50°. How many degrees in arc BC? What is the angle between secants BD and CA?

2. The angle between two secants is 20°; one of the intercepted arcs is 100°; how many degrees in the other?

165. Theorem XXVIII. *If two parallel lines intercept arcs on a circumference, the intercepted arcs are equal.*

These parallel lines may have three different positions.



Given parallel lines P_1 and P_2, cutting the circle whose center is O in points A, B and C, D respectively, intercepting arcs CA and BD.

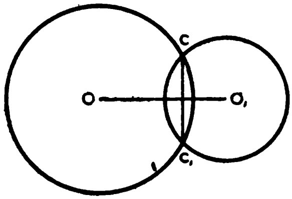
To prove that arcs CA and BD are equal.

Analysis. To prove that arcs CA and BD are equal, draw diameter HK perpendicular to P_1, and by showing that HK

is also perpendicular to P_2, show that arc CK equals arc KD, and arc AK equals arc KB.

Give the proof. Explain how the same proof answers for all three figures.

166. Theorem XXIX. *If two circumferences intersect, their line of centers is the perpendicular bisector of their common chord.*



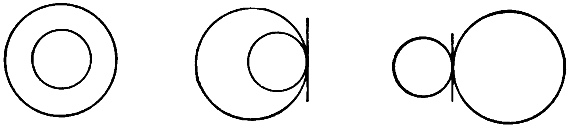
Suggestion. Suppose that the line of centers is not perpendicular to chord. Draw radii from each center perpendicular to the chord. Prove that line of centers coincides with these.

By thinking of these circles gradually moving apart, state and prove a proposition about line of centers and common tangent.

167. In the exercises below, the following definitions will be needed.

Definitions. Concentric circles are circles with a **common center.**

Two circles are **tangent** to each other when they have but one point in common; **internally tangent** when they lie on the same side of the tangent at their common point; **externally tangent** when they lie on opposite sides of that tangent line.

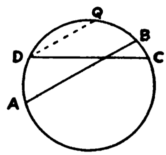


Concentric Circles. *Internally Tangent.* *Externally Tangent.*

168. Exercises.

1. Two secants are drawn from point P, the first cutting the circumference in T and R; the second in Q and S. If arc TQ is 50° and arc RS is 150°, how many degrees are there in the angle between the secants? Solve again if arc TQ is half a radian and arc RS three times as large.

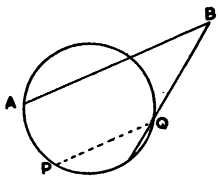
2. Let ABC be an inscribed angle with one arm BC a diameter whose center is O. OR is drawn so that arc AR equals arc RC. Prove that line OR is parallel to BA.



3. Prove Theorem XXV by the adjoining figure. Use Articles **59** and **165.**

Line QD is drawn parallel to line AB.

4. Prove Theorem XXVI by the adjoining figure.



Line PQ is drawn parallel to line AB.

5. If two tangents are drawn to a circle from the same point, prove that they are equal by drawing in the chord of tangency, and proving that two of the angles formed are equal, and hence that the triangle formed is isosceles.

6. If the angle formed by two tangents from an external point to a circle is 10°, find the size of the arc subtended by the chord of tangency.

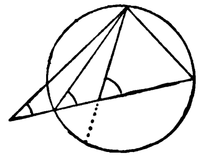
7. By inscribing an isosceles triangle in a circle, and drawing tangents at its vertices, prove a new isosceles triangle is formed.

8. An angle formed by two tangents drawn from an exterior point to a circle, is equal to one-half the difference between the central angles standing on the intercepted arcs. Show this to be a special case of Theorem XXVI.

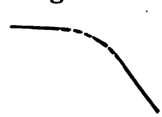
9. An angle formed by two tangents from an exterior point to a circle is equal to the supplement of the angle formed by the radii to the points of contact.

10. Can a circle be circumscribed about a rhombus?

11. If two circles are tangent, and two perpendicular secants are drawn through the point of tangency, lines joining the ends of the secants are diameters.

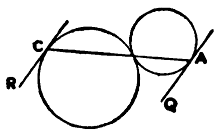


12. If three angles are drawn standing on the same chord, one with its vertex outside of the circumference, another with its vertex on the circumference, and the third with its vertex within the circle, the one with its vertex outside will be less than, and the one with its vertex inside will be greater than, the inscribed angle.

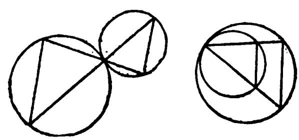


13. Two straight pieces of track, which make equal angles with the line joining their end points, are to be joined by a circular arc. To avoid jar when a car rounds the turn the parts of the track must be tangent to the arc. Show how to construct such an arc.

14. If two circles are concentric, all chords of the one, tangent to the other, are equal. Hence show that a quadrilateral which is circumscribed about the one circle and inscribed in the other must be a square.



15. If two circles are tangent externally and a secant is drawn through the point of tangency, the tangents drawn to the circles at the ends of the secant are parallel. Show that the two minor arcs, or the two major arcs, subtend equal angles at the centers of their respective circles.

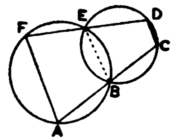


16. If two circles are tangent and two secants are drawn through the point of tangency, the lines joining the ends of the secants are parallel.

17. If two circles are tangent and three secants are drawn through the point of tangency, the lines joining their ends will form two mutually equiangular triangles.

18. An inscribed trapezoid is isosceles. Prove this. Also apply Theorems XXII and XXVIII to show that the angles at the extremities of either base are equal.

19. If two circles intersect and secants are drawn through the points of intersection, the lines joining the ends of the secants are parallel.



Suggestion. Draw the common chord and prove that the angle A and angle C are supplementary by proving that they are respectively supplementary to angles at E.

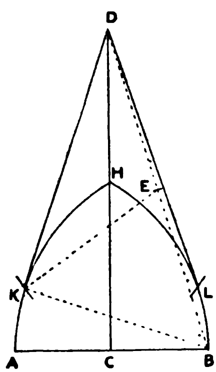
20. Prove that the sum of the angles of a triangle is equal to a straight angle by inscribing the triangle in a circle and drawing a tangent to the circles at one of the vertices.

21. If a hexagon is circumscribed about a circle, the sum of the first, third and fifth sides is equal to the sum of the second, fourth and sixth sides.

22. Is it true for any circumscribed polygon that the sum of one alternate set of sides is equal to the sum of the other alternate set of sides?

23. By applying Theorem XXII, show that if an inscribed polygon is equi-lateral, it is also equi-angular. Also prove the converse. Prove the same for a circumscribed polygon.

24. Construct a Gothic arch and construct tangent lines from a point directly over the center of the arch.

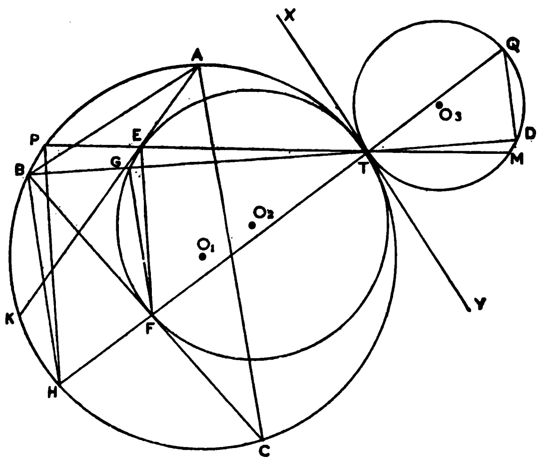


If the arch is 12 feet wide, and the tangents are drawn from a point 20 feet above the middle of the base, find the length of each tangent line.

Suggestion. To construct tangent line DK, draw a circle on BD as diameter and let this circle cut arc AH at K. Then DK will be tangent to arc AH because $\angle BKD = 90°$. To calculate DK, use rt. $\triangle BDK$, in which you can easily find BK and BD.

25. In the following figure, the circles O_1, O_2, O_3 are all tangent to the line XY at the point T. AK and BC are chords of circle whose center is O_1 and tangent to the circle whose center is O_2 at the points E and F respectively. Secants TE, TG, TF are drawn, cutting the circumference whose center is O_1 in the points P, B, H respectively, and the circumference whose center is O_3 in the points M, D, Q, respectively. Lines are drawn joining these points. Also lines EG, GF, EF, AB, AC, are drawn.

Name all the pairs of equal arcs that you can find in the figure, and tell reason for knowing them to be equal. Name all the pairs of equal angles that you can find, telling reason for knowing them equal. A pair of equal arcs that is apt to be overlooked is BH and HC.



169. Summary of Chapter IV.

Part I. Central Angles, Arcs, Chords.

Definitions. Circle, radius, diameter, chord, equal circles, arc, semicircle, semicircumference, sector, central angle.

Postulates.

1. A circumference is determined when its center and radius are known.

2. A point is within, on, or without a circle according as its distance from the center is less than equal to or more than the radius.

Propositions on central angles and arcs.

Theorem I. In the same or equal circles if two central angles are equal, the arcs on which they stand are equal.

Corollary 1. A diameter divides a circumference into two equal parts.

Corollary 2. In the same or equal circles sectors which have equal angles are equal.

Corollary 3. A diameter divides a circle into equal parts.

Theorem II. In the same or equal circles if two central angles stand on equal arcs, the angles are equal.

Definition of measurement of arcs.

Theorem III. In the same or equal circles if two central angles are unequal, the arcs on which they stand are unequal and the greater angle stands on the greater arc.

Theorem IV. In the same or equal circles if central angles stand on unequal arcs, the angles are unequal, the greater angle standing on the greater arc.

Functions. Definition and Illustrations.

Theorems on Arcs and Chords.

Theorem V. In the same or equal circles if two arcs are equal they are subtended by equal chords.

Theorem VI. In the same or equal circles if two chords are equal they subtend equal arcs.

Theorem VII. In the same or equal circles if two minor arcs are unequal they are subtended by unequal chords, and the greater arc is subtended by the greater chord.

Theorem VIII. In the same or equal circles if two chords are unequal, they subtend unequal minor arcs, and the less chord subtends the less minor arc.

Summary of the above illustrating functions, independent and dependent variables.

Theorem IX. A diameter which is perpendicular to a chord bisects the chord and its subtended arc.

Theorem X. A diameter which bisects a chord is perpendicular to it.

Theorem XI. The perpendicular bisector of a chord passes through the center of the circle and bisects the subtended arc.

Corollary. In general two chords cannot bisect each other.

Theorems on Chords and Distances from the Center.

Theorem XII. In the same or equal circles equal chords are equidistant from the center.

Theorem XIII. In the same or equal circles if two chords are equidistant from the center, they are equal.

Theorem XIV. In the same or equal circles if two chords are unequal, they are unequally distant from the center; the greater chord is the less distant.

Theorem XV. In the same or equal circles if two chords are unequally distant from the center, they are unequal; the chord less distant is the greater.

Corollary. The diameter is the longest chord in a circle.

Summary of the above, illustrating functions, independent and dependent variables.

Propositions on Tangents.

Theorem XVI. Of all lines drawn through a point on the circumference, the perpendicular to the radius drawn to that point is the only one that does not meet the circumference again.

Theorem XVII. One and only one tangent can be drawn to a circle at a point on its circumference.

Theorem XVIII. A tangent is perpendicular to a radius drawn to the point of tangency.

Theorem XIX. A line perpendicular to a radius at its extremity on the circumference is tangent to the circle.

Theorem XX. The center of a circle lies on the perpendicular to a tangent at the point of tangency.

Theorem XXI. The perpendicular from the center of a circle to a tangent meets it at the point of tangency.

Part II. Inscribed Angles.

Definitions. Segments of a circle. Inscribed angle.

Theorem XXII. An inscribed angle is equal to half the central angle standing on the same arc.

Corollary. Angles inscribed in the same or equal segments are equal.

Theorem XXIII. An angle inscribed in a segment is greater than, equal to or less than a right angle, according as the segment is less than, equal to, or greater than a semicircle.

Definition of an inscribed polygon.

Problem I. To construct a tangent to a circle at a given point on its circumference.

Problem II. To construct a tangent to a circle that shall pass through a given external point.

Theorem XXIV. An angle formed by a tangent and a chord is equal to half the central angle standing on the intercepted arc.

Corollary. Tangents to a circle from the same point are equal.

Theorem XXV. An angle formed by two chords intersecting within a circle, is equal to half the sum of the central angles standing on the intercepted arcs.

Theorem XXVI. An angle formed by two secants intersecting without the circle is equal to half the difference between the central angles standing on the intercepted arcs.

Theorem XXVII. An angle formed by a secant and a tangent is equal to half the difference between the central angles standing on the intercepted arcs.

Theorem XXVIII. If two parallel lines intercept arcs on a circumference, the arcs are equal.

Theorem XXIX. If two circumferences intersect, their line of centers is the perpendicular bisector of their common chord.

1. Into what two parts is the subject matter of this chapter divided?

2. Is the subject matter of the different parts closely related?

3. Examine the theorems of this chapter. Select some that have been proved without one or more of the preceding chapters? Explain your selection by giving an outline of the proof.

4. Are there any theorems of the second part that have no dependence on those ot the first part?

5. Of the solids which you examined in Chapter I, which do you think might be so cut by a plane that the section would be circular?

6. State some of the exercises that you consider to have some connection with real things around you.

The theorems of this chapter are selected from Book III of Euclid's Geometry.

CHAPTER V

PART I—LOCI OF POINTS. PART II—COÖRDINATE GEOMETRY.

PART I—LOCI OF POINTS

170. General Discussion. In order to give the location of a place it is necessary to tell its position with reference to some other places whose locations are known to both the person seeking the information and the person giving it. In fact it will be found that in order to state definitely the location of any place, three distinct items of information concerning that place must be mentioned.

For example, I have in mind a certain point in this room. Can you tell me the exact location of the point? It is two feet from the north wall. Can you now determine it more definitely than you could before? I say that it is four feet from the ceiling. Can you now tell the exact location of the point? I give the further information that it is equidistant from the east and west walls. What is the exact location of the point?

Again, suppose that you wish to tell the location of the town in which you live. You state that its latitude is so many degrees and that its longitude is so many degrees, say, 41 degrees north latitude and 73 degrees west longitude. This description involves the knowledge of an imaginary line called the equator from which to count the latitude 41 degrees north, and the knowledge of an imaginary line north and south through the town of Greenwich, from which to count the longitude 73 degrees west. There is the further information that the town is on the earth's surface, which is a third fact, telling us the distance that the town is from the center of the earth.

To locate the position of an observatory situated at the top of a mountain, we would give its latitude, its longitude, and its height above sea level.

171. Locus. It is seen readily that when but one fact is mentioned in our description, there exist a great many points that answer the description. In the illustration of the location of a point in the room, there is a whole plane of points that answer the description "two feet from the north wall." After we added to our description "four feet from the ceiling," there was still left a whole line of points answering the given descriptions.

Definition. The line or group of lines, the plane or group of planes, every point of which satisfies certain given conditions, and on which every point that satisfies these conditions lies, is called the *locus of the point that satisfies these conditions.*

Whenever we wish to prove that a line is the locus of a point satisfying a given condition, we must prove two distinct things.

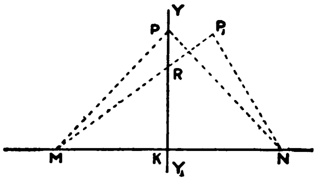
First, every point on the line satisfies the given condition. Second, every point satisfying the given condition is on the line. The one of these is just as important as the other.

We shall now take up some theorems involving a few simple loci.

172. Theorem I. *The locus of a point at a given distance from a given fixed point is a circumference about the given fixed point as a center, with the given distance as a radius.*

This follows immediately from the definition of circumference, Art. **122,** and Postulate VIII, Art. **123,** and might have been given as a corollary to the statements there made.

173. Theorem II. *The locus of a point equidistant from two fixed points is the perpendicular bisector of the line that joins them.*



Given the two fixed points M and N. Also line YY_1 the perpendicular bisector of line MN.

To prove that line YY_1 is the locus of a point equidistant from M, N.

Analysis. In order to prove this we shall prove that P, any point on YY_1, is equidistant from M and N, and that P_1, any point not on YY_1, is not equidistant from M and N. In other words we shall prove that PM equals PN, and that P_1M is greater than P_1N.

We can prove PM equal to PN if we can prove triangles MKP and KNP congruent. This the student can readily do.

We can prove P_1M greater than P_1N by drawing the line RN and noticing that $NR + RP_1$, which is equal to $MR + RP_1$, is greater than P_1N. (See Art. **44**.)

Give these proofs and state theorem proved.

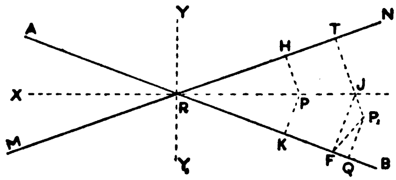
Exercises.

1. Apply this theorem to show that the perpendicular bisector of a chord of a circle is a diameter.

2. Show that either diagonal of a rhombus, produced, is the locus of points equidistant from two of the vertices.

3. What is the locus of the vertex of an isosceles triangle, if the base is fixed while the length of the equal sides varies?

174. Theorem III. *The locus of a point equidistant from two lines is the bisectors of the angles formed by those lines.*



Given the lines AB and MN, and the lines YY_1 and XX_1 bisecting the angles NRA, MRB and BRN, ARM respectively.

To prove that YY_1 and XX_1 form the locus of points equidistant from AB and MN.

Analysis. To prove this let P be any point on either YY_1 or XX_1, and P_1 be any point not on these lines. We must prove that point P is equidistant from AB and MN, and that P_1 is not equidistant from AB and MN. Draw the perpendiculars PK and P_1Q to AB, and perpendiculars PH and P_1T to MN. The lines PK and PH, which represent the distances of P from the line AB and MN, respectively, can be proved equal, if we can prove triangles RKP and RPH congruent. The student can readily prove this. We must now prove that P_1Q does not equal P_1T. Since P_1 is not on the bisectors, P_1Q or P_1T will be cut by one of the bisectors. Let XX_1 cut P_1T at point J. Draw $JF \perp AB$. Draw P_1F. Since JT is equal to JF, we can prove P_1T greater than P_1Q if we can prove that $P_1J + JF$ is greater than P_1F, which in turn is greater than P_1Q.

Proof.	$P_1F > P_1Q.$	Why?
	$P_1J + JF > P_1F.$	Why?
	$JT = JF.$	Why?
$\therefore$	$P_1J + JT > P_1F.$	Why?

$\therefore$ $P_1J + JT > P_1Q$. Why?

That is $P_1T > P_1Q$. Why?

State the theorem that we have just proved.

175. Theorem IV. *The locus of a point equidistant from two given parallel lines is the parallel line mid-way between them.*

Let the student give figure, analysis, and proof.

Note. In the figure of Theorem III imagine the points H and K remaining fixed while the point of intersection R moves off farther and farther to a very great distance to the left. Line YY_1 moves with R, but line XX_1 keeps its position, while the lines BA and MN approach the condition of being parallel to XX_1.

176. Theorem V. *The locus of a point at a given distance from a given line consists of two lines parallel to the given line, one on either side of it, and at the given distance from it.*

Prove by marking a point on one of the parallel lines and showing that it is at the given distance from the given line, then marking a point not on one of the parallels and showing that it is not at the given distance from the given line.

Exercises.

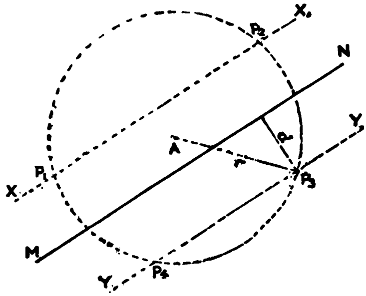
1. Draw two intersecting lines, and construct the locus of points equidistant from these lines.

2. Draw two parallel lines three inches apart. Construct the locus of points equidistant from these lines.

3. Construct the locus of points one inch from a given line. Do this by drawing two perpendiculars to the given line, and marking on them points one inch from the given line.

177. In the following it will be taken for granted that the work is all in the same plane. Then only two conditions are necessary to fix a point completely. If only one condition be given, the point is left free to move and describe a locus. To make this clear consider the following problem.

178. Problem in Loci. *To find a point r units distant from a given point, and d units distant from a given line.*



We have given as standards of reference:

1st. point A,

2nd. line MN.

We have given these conditions to be satisfied:

1st. The point we are locating is r units from point A.

2nd. The point we are locating is d units from line MN.

Analysis. The locus of a point which satisfies the first condition is a circumference about A as a center, with a radius r. Why?

The locus of a point which satisfies the second condition, is two lines one on each side of line MN at the distance d from MN. Call these lines XX_1 and YY_1.

Where circumference C cuts the lines XX_1 and YY_1, P_1, P_2, P_3, P_4, are the required points. Each one of these points satisfies the required conditions.

Let the student give construction and proof.

Discussion. There is nothing in the problem stating the relative position of the point A and the line MN. They can be placed as near together or as far apart as we choose. Also r and d may be any lengths. Make drawings placing line MN at different distances from point A, leaving r and d the same, and discover that there may be one, two, three, four points, or no point that satisfies these conditions.

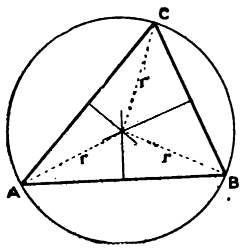
179. Problems and Theorems. According to the illustration above solve the following problems.

1. A railroad and a county road cross. A station is on the railroad. A man wished to place a saw-mill equally distant from the railroad and the county road, and a quarter of a mile from the station. Find the possibilities of such a location.

2. In placing a statue on a public square, it is desired to have it equidistant from two sides opposite each other, and ten feet from a third side. Find possibility of its location.

3. A woman wished to plant a shrub equidistant from the corner of her house and the corner of her lot, and at the same time equidistant from the front line of her house and the front line of her lot. Find the possibilities of its location.

4. Draw a triangle ABC. Construct the perpendicular bisector of side AB. Of what points is this the locus? Construct the perpendicular bisector of side BC. Of what points is this the locus? Do these two bisectors intersect? Why?

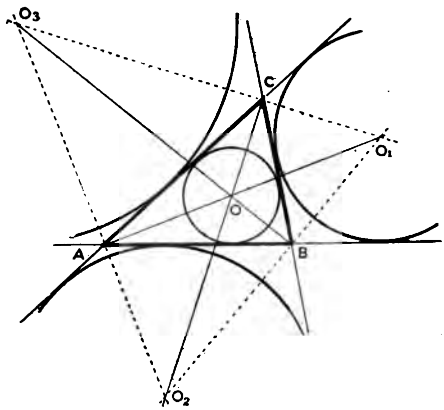


How does this point of intersection lie with reference to the points C and A? Therefore, in what line must it lie? Why? What can you now say about the perpendicular bisectors of the sides of a triangle? What circle can you describe with the point of intersection as a center?

The result of this exercise may be stated:

Theorem VI. *The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from its vertices. This point is the center of the circumscribed circle.*

5. Draw a triangle ABC. Construct the locus of points equidistant from sides AB and AC. (Do not forget that it requires two lines for this locus.) Construct the locus of a point equidistant from sides BA and BC. Do these lines intersect? Why? How many points of intersection are there? How do these points lie with reference to the lines CA and CB? Why? On what other lines do these points fall? What can you say about the bisectors of the angles of a triangle?

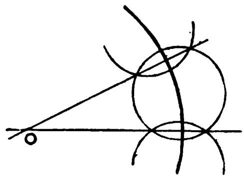


Theorem VII. *The bisectors of the angles of a triangle meet in a point equidistant from its sides. This point is the center of the inscribed circle.*

If the exterior angles are bisected also, three other points equidistant from the sides are located. These are the centers of the escribed circles.

6. To find the center of a given circle. (Use Theorem VI.) It will of course be sufficient to draw perpendicular bisectors of any two sides of the inscribed triangle, or of any two chords. Draw a circle with a radius of two inches and erect perpendicular bisectors of several chords.

7. To find the center of a given arc. Draw three intersecting circles whose centers are on the given arc, and draw their common chords. They will intersect at the center of the arc. Prove.



8. A design for wall paper is a circle inscribed in an equilateral triangle. Make such a design, filling a square piece of paper.

9. Make a design in which a circle circumscribes an equilateral triangle.

10. Make a design in which a circle circumscribes a square.

180. Exercises. Location of circles that must fulfill given conditions.

When called upon to find the locus of a point that fulfills certain conditions, it is well to locate several points that fulfill the condition and draw a line through them. If you can show that every point on the line thus determined fulfills the condition and that there exists no other point that does do so, then the line is the correct locus.

The student is warned against failure to get all the lines when there is more than one, as in the case of the locus of a point equidistant from two intersecting lines or the locus of a point at a given distance from a given line.

1. What is the locus of the center of a circle whose radius is r:

(a) that will touch a given straight line MN?

(b) that will touch a given circle C?

(c) that will pass through a given point P?

2. Construct a circle that will touch a given line MN, and pass through a given point P. Draw several such circles and study the locus of their centers.

3. Construct a circle with given radius that will touch a given line MN and a given circle C.

4. What is the locus of the center of a circle that will touch two given intersecting lines?

5. Construct the locus of the center of a circle that will touch two given parallel lines.

6. What is the locus of the center of a circle that will touch two given concentric circles?

7. What is the locus of the center of a circle that will pass through two given points?

8. Construct a circle that will pass through three given points not all in the same straight line.

9. Construct a circle with given radius, tangent to two given circles.

10. Locate a point which is at a given distance from two intersecting circles. Is there more than one such point?

11. Cut a circle of two inches radius out of one corner of a sheet of paper without wasting any more paper than necessary.

12. With a given radius, say half an inch, construct as many circles as possible on an ordinary sheet of paper, without overlapping.

13. Construct three equal circles within an equilateral triangle that will be tangent to one another and to the sides of the triangle. (Each circle is tangent to two sides.) This is a common design.

14. What is the locus of the center of a circle that will be tangent to a given line at a given point?

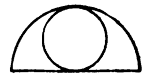
15. What is the locus of the center of a circle that will be tangent to a given circle at a given point?

Suggestion. Draw a tangent at the given point and find the locus of the center of the circle that will be tangent to this tangent at the given point.

16. Construct a circle that will be tangent to the line AB at the point A, and pass through a point C not on the line.

17. In Exercise 16 how will any angle inscribed in the segments of the circle constructed compare with the angle CAB?

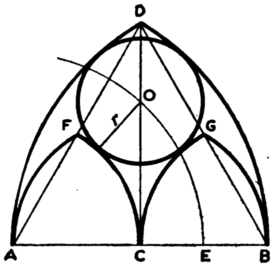
18. Construct a circle that will be tangent to a given semicircle and its diameter. This is a common design.



19. Construct a circle which shall be tangent to one arm of a given angle at the vertex and cut a given segment from the other arm.

20. If two circles intersect, draw two circles that will be tangent to both of them, one tangent internally and the other externally.

21. The adjoining figure consists of three Gothic arches and a tangent circle.



Construct such a figure taking $AC = CB = 2$ inches.

Suggestion. To draw the tangent circle first locate the point O as follows:

(a) Point O must lie on line CD.

(b) Point O must lie on arc EO which has the same center as arcs CF and BD, and is mid-way between them.

Show that $r = \frac{1}{4} AB$.

22. What is the locus of the vetex of a triangle having a fixed base and a given altitude?

23. What is the locus of the vertex of a triangle having a fixed base and a given median to the base?

24. What is the locus of the vertex of a triangle having a given base and a given area?

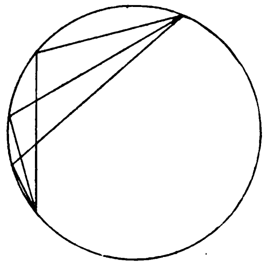
25. What is the locus of the vertex of a triangle having a given base and a given angle opposite it?

26. Construct a triangle having a given base, a given altitude, and a given angle at the base.

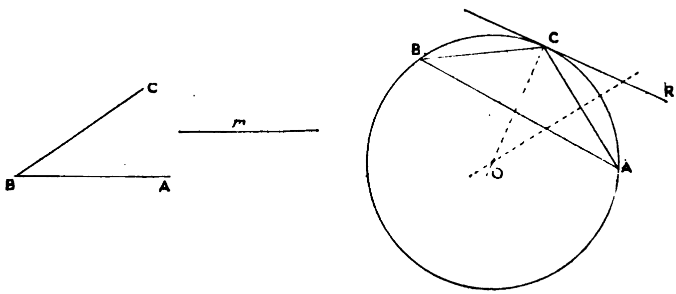
27. What is the locus of the mid-points of chords drawn parallel to each other in a given circle?

28. What is the locus of the mid-points of chords drawn through a given point on a given circumference?

181. Theorem VIII. *The locus of the vertex of a given angle whose arms pass through two given fixed points is the arc capable of containing the angle and passing through the given points.* See Art. **155,** Corollary.



182. Problem I. *To construct an arc on a given line as a chord, capable of containing a given angle.*



Given the angle ABC **and the segment** m.

To construct an arc on m as a chord capable of containing angle ABC.

Analysis. Suppose that the work is done (see figure) and the chord AC is equal to the given line m and angle ABC is equal to the given angle ABC.

Examining the figure we find that if $\angle$ ACR is formed by drawing tangent CR, $\angle$ ACR = $\angle$ ABC. Why? Then if we construct an arc of which AC is the chord and CR is a tangent making $\angle ACR = \angle ABC$, this arc will contain $\angle ABC$, and is the arc required.

Therefore our problem resolves itself into the problem of constructing an arc which shall have AC as a chord and CR as a tangent.

This in turn becomes the problem of finding the center of a circle, which shall pass through the given points A and C and touch the given line CR at the point C, $\angle ACR$ having been made equal to the given angle ABC.

The locus of the center is

1st. the perpendicular bisector of the line joining A and C.

2nd. the line perpendicular to the line CR at the point C.

The point of intersection of these two loci is the point required, that is, the center of the arc whose chord is line m and capable of containing angle ABC.

Make this construction and prove correctness of the process.

183. Exercises.

1. Construct a triangle having given the base, the altitude, and the angle opposite the base.

2. Construct a triangle having given the base, the angle opposite the base, and the median to the middle of the base.

3. Construct a triangle having given the base, altitude, and median to the base.

4. Given the hypotenuse of a right triangle, what is the locus of the vertex of the right angle?

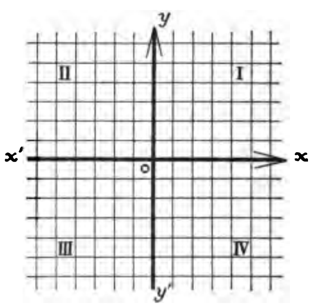
5. If several triangles all have the same base, and equal angles opposite the base, prove that the bisectors of these angles all pass through the same point.

PART II—COÖRDINATE GEOMETRY

184. Coördinates. We shall now study a very convenient method for fixing positions on a plane, and shall then make use of the method to demonstrate some familiar problems. We shall also add some new theorems and problems.

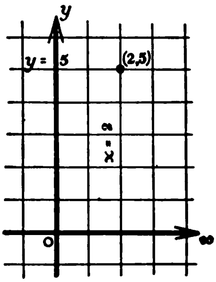
To locate the position of a point on a sheet of paper or any plane surface, we follow the same plan as is used to locate a place on the earth's surface. Draw two lines of reference which cut at right angles; one extending from left to right and the other up and down. (Always use cross-section paper for this work.)

The plane of the paper is now divided into four quarters, called quadrants. They are numbered I, II, III, IV as shown in the figure. We now locate a point by giving its distance from the two reference lines, stating also whether these distances are measured to the right or left, upward or downward. The line $x'x$, as shown in the figure, is called the *axis* of the *abscissa.* The line $y'y$ is called the *axis* of the *ordinate.* Distances upward from the axis of the abscissa are positive; downward, negative. Distances to the right of the axis of the ordinate are positive; to the left, negative. The point of intersection marked O is called the *origin.* The distance which a point is to the right or left of the axis of the ordinate, measured in the direction of the axis of the abscissa, is called the *abscissa* of the point. The distance which a point is from the axis of the abscissa, measured in the direction of the axis of the ordinate, is called the *ordinate* of the point. The two together are called the *coördinates* of the point which is fixed by them.



We usually represent the abscissa of a point by x, and its ordinate by y; then the two numbers x, y are the coördinates of the point. However, any other letters may be used.

Example. Suppose we wish to locate a point for which $x = 2$ and $y = 5$, that is, the abscissa is 2 and the ordinate is 5.



Since the abscissa is 2, the point lies 2 units to the right of the axis of the ordinate. It is therefore on a line drawn 2 units to the right of the axis of the ordinate and parallel to that axis. This line we designate by the equation $x = 2$, because this equation is true for any point on the line, and is true for no other point. This line is the *locus* of the point whose abscissa is 2.

Since the ordinate of the point is 5, the point must be somewhere on a line drawn parallel to the axis of the abscissa and 5 units above it. This line is designated by the equation $y = 5$. Why? This line is the *locus* of a point whose ordinate is 5.

The required point must be at the intersection of the loci just drawn. We call it the point (2, 5); here we inclose in parentheses the coördinate of the point, writing first the abscissa, and then the ordinate, with a comma between.

A point whose abscissa is x and whose ordinate is y is designated by the symbol (x, y).

185. Exercises. Draw the lines on which each of the following points must lie. On each line write the equation which describes it. As has just been explained such a line is the locus of the point which lies on it.

1. What is the locus of a point whose abscissa is 3? Whose ordinate is 7? What point is at the intersection of the two loci? Draw a complete figure as shown above, marking in the equations of the lines and the symbol for their point of intersection.

2. Proceed as in Exercise 1, when the abscissa is 3 and the ordinate is -7.

3. Proceed as in Exercise 1, when the abscissa is -5 and the ordinate is 2.

4. Proceed as in Exercise 1, when the abscissa is − 5 and the ordinate is − 2.

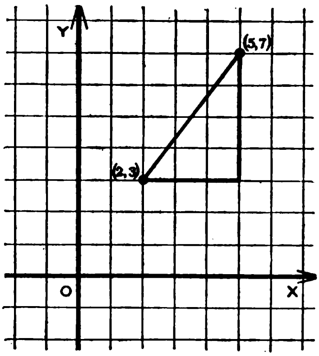
5. Proceed as in Exercise 1, when the abscissa is − 3 and the ordinate is − 3.

6. Make diagrams as in Exercise 1 for each of the following points:

(2, 6); (− 5, 4); (6, − 9); (12, − 7); (− 2, − 3); (0, − 9); (2, 0).

7. What line is described by the equation $x = 0$? What line is described by the equation $y = 0$? What point lies at the intersection of these loci?

8. Locate the points (9, 4); (− 3, − 1); (4, − 3). Join by straight lines. What kind of a figure is formed?



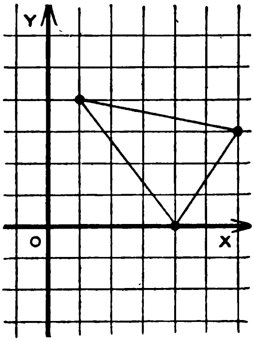
186. The Distance between two Given Points. We wish to express the distance between two given points when the coördinates of these points are known. For this we locate the points as in the adjoining figure and by counting and using the Pythagorean Theorem we find the distance.

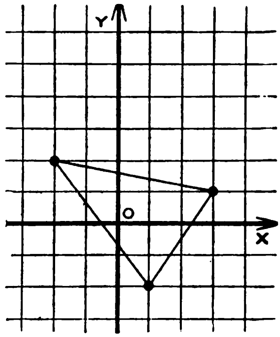
187. Exercises.

1. Determine the distance between each of the following pairs of points:

(a) (1, 2), (4, 6).
(b) (3, 2), (7, 5).
(c) (2, 1), (4, 3).
(d) (− 2, 1), (1, 3).
(e) (− 33, −2), (1, 1).
(f) (− 6, 2), (− 6, 7).
(g) (3, − 1), (− 1, − 3).
(h) (5, 2), (− 7, − 3).

2. Determine the lengths of the sides of the triangles shown in each of the following figures.



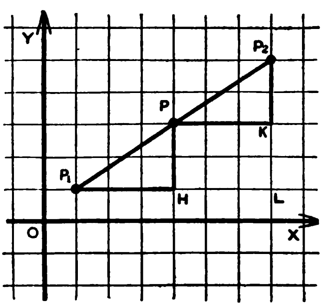


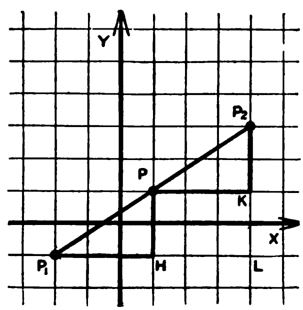
3. Show that the points (2, 2), (4, 6), and (– 3, 7) form an isosceles triangle. Draw an accurate figure.

4. Show that the distance from the origin (0, 0) to any point (a, b) is $\sqrt{a^2 + b^2}$.

5. Draw the quadrilateral whose vertices are (6, 6), (– 2, 4), (– 6, – 2) and (– 4, – 4). Calculate the lengths of its sides and its diagonals.

188. The Point Mid-way between Two Given Points. Study each of the following figures to determine the position of point P, which is mid-way between the given points P_1 and P_2.





2. Locate points (9, 6), (− 2, 3), (− 4, − 4), (7, − 1). Join in order by straight lines. Show that the opposite sides of the figure have equal slopes, and hence are parallel. What kind of a figure is formed?

3. Locate the points (5, 4), (− 3, 2), (1, − 6) and join them by straight lines, forming a triangle.

Determine the coördinates of the mid-points of any two sides of the triangle. Then show that the line joining these mid-points is parallel to the third side.

4. Construct a quadrilateral whose vertices are the points (4, 7), (− 2, 5), (− 6, − 3), (6, − 5). Find the coördinates of the mid-points of the four sides. Show that the mid-points are the vertices of a parallelogram, by showing that the opposite sides of the figure formed by them are parallel.

192. Summary of Chapter V. Loci of Points.

Part I—Loci in General.

General Discussion.

Theorem I. The locus of a point at a given distance from a given fixed point is the circumference described about the fixed point as a center, with a radius equal to the given radius.

Theorem II. The locus of a point equidistant from two given fixed points is the perpendicular bisector of the line joining them.

Theorem III. The locus of a point equidistant from two given intersecting lines consists of the bisectors of their included angles.

Theorem IV. The locus of a point equidistant from two given parallel lines is a line parallel to them and mid-way between them.

Theorem V. The locus of a point at a given distance from a given line consists of a pair of parallels one on each side of the given line at the given distance from it.

Locations of points by means of loci.

Discussion and Examples.

Theorem VI. The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices of the triangle.

Theorem VII. The bisectors of the angles of a triangle meet in a point which is equidistant from the sides of the triangle.

Theorem VIII. The locus of the vertex of a given angle whose arms pass through two fixed points, is the arc capable of containing the angle, and passing through the given points.

Problem I. On a given line segment as a chord to construct the segment of a circle capable of containing a given angle.

Part II—Coördinate Geometry.

Definitions. Location of points on a plane.

Finding distances between two points.

Finding a point mid-way between two points.

Finding the slope of a line between two points.

1. Into what two parts is this chapter divided?

2. What is meant by the locus of a point to fulfil a given condition?

3. If you wish to locate a point in a plane how many conditions must be given? Hence how many loci will be necessary to fix the point?

4. When you have given the abscissa and ordinate of a point, have you two conditions for its location given? What are the loci that correspond to them?

5. Given the abscissas and the ordinates of two points, how do you find the length of the line joining them? How do you find the midpoint of this line?

6. Have any of the theorems of this chapter been proved without any of the preceding chapters? If so, state them.

7. With what preceding theorems are the theorems of this chapter most closely connected?

Historical Note. The subject of Coördinate Geometry is often termed "Cartesian" Geometry, after the name of its founder, Descartes (1596–1650). He introduced the idea of representing the position of a point by means of coördinates, and the locus of a point by means of an equation between those coördinates. The resulting union of algebra and geometry has been a powerful aid in the development of both subjects, and constitutes one of the great steps in the development of mathematical science.

CHAPTER VI

PART I—RATIO AND PROPORTION. PART II—SIMILAR FIGURES. PART III—TRIGONOMETRIC FUNCTIONS

PART I—RATIO AND PROPORTION

193. Definitions and Notation. We have been dealing with the subject of ratio from the beginning of our course. We have defined the **ratio** between two numbers as the number by which the second must be multiplied to obtain the first.

If $a \div b = r$, then r is the ratio of a to b, meaning that $br = a$. This ratio is often indicated by the symbol $a : b$, or by $\frac{a}{b}$, in place of $a \div b$.

If four numbers are so related, that we must multiply the fourth by the same number to obtain the third that we must multiply the second by to obtain the first, the four numbers are said to form a **proportion.** In other words a *proportion is the equality of two ratios.*

Thus: $\frac{a}{b} = \frac{c}{d}$, or $a \div b = c \div d$, or $a : b = c : d$, are symbols for the equality of ratios, or proportions.

The equality of more than two ratios is called a **continued proportion.** Thus:

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots\cdots; \text{ or } a : b = c : d = e : f = \cdots\cdots.$$

Since numbers may be represented by line-segments, the subject of ratio and proportion is treated both geometrically and algebraically.

We note the following correspondence of terms:

Geometry	*Algebra*
Line-segment	Number
Commensurable	Rational
Incommensurable	Irrational

Line-segment is the portion of a line between two definitely located points on a line.

Number expresses the ratio which such a segment bears to another line-segment selected as a unit of measure.

Determining the ratio is equivalent to measuring the line. The expression of this ratio is number.

We have seen that some line-segments exist for which a common unit of measure can be found. These line-segments are said to be *commensurable*. Numbers which express their ratios are called *rational numbers*.

Other line-segments, such as the diagonal and the side of a square, the circumference and the diameter of a circle, have no common unit of measure. We call them *incommensurable* lines. Numbers which express their ratios are called *irrational*.

The radical sign and the fractional exponent have been introduced in the expression of these ratios. (Review First Course, page 172.)

In *practical* work all numbers are treated as commensurable, since we can make the unit of measure as small as we please, and so have the work as nearly accurate as necessary.

194. Definitions.

The terms of a ratio are named **antecedent** and **consequent** respectively.

Thus, if r is the ratio of a to b, $r = a : b$, then a is the antecedent and b is the consequent. These correspond to the numerator and the denominator of a fraction, the dividend and the divisor of a division.

In a proportion the first and last terms are called **extreme** terms, and the second and third terms are called **mean** terms.

Thus in the proportion $2 : 3 = 4 : 6$, 2 and 6 are extremes and 3 and 4 are means.

When a mean term is repeated, it is called a **mean proportional.**

Thus in the proportion $2 : 10 = 10 : 50$, 10 is the mean proportional between 2 and 50.

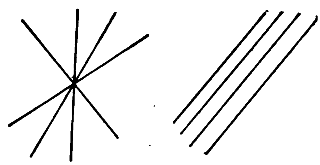
Exercise. Give several pairs of numbers which can be used as means when the extremes are 4 and 16. What is the mean proportional between these numbers?

195. Geometric Illustrations of Ratio and Proportion.

Since a rigid treatment of the theory of ratio and proportion is too difficult for beginners, we shall regard the following geometric treatment more as illustrative, and thus show the meaning of the algebra.

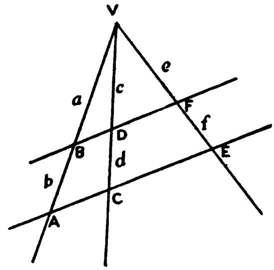
Definitions.

If through a point any number of lines are drawn, the figure so formed is called a **pencil of lines.** The point is called the **vertex** of the pencil.



A number of lines parallel to one another is called a **pencil of parallels.**

Exercise. Draw a pencil of three straight lines cut by two parallel lines as in the figure.



Measure a, b, c, d, e, f.

Calculate the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$. Divide out to two decimal places.

Do you find any relation between these ratios?

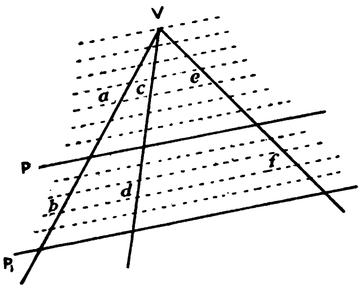
Calculate the ratios $\frac{b}{a}, \frac{d}{c}, \frac{f}{e}$.

Do you find a relation between these ratios?

If you divide 1 by each of the ratios of the first set, you will obtain the ratios of the second set. The ratios of one set are said to be the reciprocals of the corresponding ratios of the other.

The truth brought out by your measurements is expressed by the following theorem.

196. Theorem I. *If a pencil of lines is cut by a pencil of parallels the corresponding segments are in proportion.*



Given the pencil of line whose vertex is V, cut by the parallels P and P_1, cutting the segments a, b, c, d, e, f, respectively.

To prove that $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

Proof. Assuming that the segments a and b are commensurable, let u be a unit of measure common to them. Let m be the number of times that u is used to measure a, and m' the number of times that u is used to measure b. Through the points of division on a and b, draw lines parallel to P. These lines will be parallel to P_1 also. Why?

These lines will divide c and e into m equal parts, and d and f into m' equal parts? Why? Each part of c and d is, say, u_1 units long, and each part of e and f is u_2 units long.

So we have $a = mu, \quad b = m'u;$

$c = mu_1, \quad d = m'u_1;$

$e = mu_2, \quad f = m'u_2.$

Forming our ratios, we have:

$$\frac{a}{b} = \frac{mu}{m'u} = \frac{m}{m'};$$

$$\frac{c}{d} = \frac{mu_1}{m'u_1} = \frac{m}{m'};$$

$$\frac{e}{f} = \frac{mu_2}{m'u_2} = \frac{m}{m'}. \qquad \text{Why?}$$

Therefore $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f}. \qquad \text{Why?}$$

State the proposition proved.

Remark. Notice that we assumed a and b to be commensurable lines. But a might be the side of a square of which b is the diagonal, in which case the proof given above would not answer. However, as we said before, we shall assume our propositions hold for incommensurable cases. These will be discussed later when the theory of limits has been studied. Art. **289.**

The proof has been given for three lines of the pencil, leading to three equal ratios. Evidently any number of lines could be used.

197. Definition. The **reciprocal** of a number is 1 ÷ by that number: thus, the reciprocal of 2 is $1 \div 2$ or $\frac{1}{2}$, of $\frac{2}{3}$ is $1 \div \frac{2}{3}$ or $\frac{3}{2}$, of $\frac{a}{b}$ is $1 \div \frac{a}{b}$ or $\frac{b}{a}$.

By inverting the ratios used in Theorem I we could have shown just as well that

$$\frac{b}{a} = \frac{d}{c} = \frac{f}{e}.$$

The relation of this result to the result of Theorem I is usually stated thus:

198. Theorem II. *If quantities are in proportion, they are in proportion by inversion.*

199. Theorem III. *If four quantities are in proportion, they are in proportion by alternation, that is, the first is to the third as the second is to the fourth.*

Geometric proof:

Taking the figure and notation of Theorem I, by the same plan of proof, show that $\frac{a}{c} = \frac{b}{d}$.

Algebraic proof:

We have $$\frac{a}{b} = \frac{c}{d}.$$

Multiply both sides of the equation by $\frac{b}{c}$;

Therefore $$\frac{a}{c} = \frac{b}{d}.$$ Why?

200. Exercise. In your figure for the preceding exercise (Art. **195**) measure VA, VC, VE, BA, DC, FE. Find the ratios $\frac{VA}{BA}, \frac{VC}{DC}, \frac{VE}{FE}$.

How do these ratios compare with one another? How do they compare with those of the preceding exercise? Explain from the figure why each ratio of this set exceeds the corresponding ratio in the first set by 1.

Using the notation of Theorem I, $VA = a + b$, $VC = c + d$, $VE = e + f$; therefore these equal ratios may be written

$$\frac{a+b}{b} = \frac{c+d}{d} = \frac{e+f}{f}.$$

Apply this to each of the following proportions:

(*a*) $3:5 = 6:10 = 12:20$;

(*b*) $6:4 = 30:20 = 12:8 = 3:2$.

Expressed as a theorem, we have

201. Theorem IV. *In a series of equal ratios, the sum of the first antecedent and first consequent is to the first consequent as the sum of the second antecedent and second consequent is to the second consequent, as the sum of the third antecedent and third consequent is to the third consequent, and so on.*

Geometric Proof.

Using the figure and notation of Theorem I, we have

$$a + b = mu + m'u, = (m + m')\,u;$$
$$c + d = mu_1 + m'u_1, = (m + m')\,u_1;$$
$$e + f = mu_2 + m'u_2, = (m + m')\,u_2.$$

Then our ratios are

$$\frac{a+b}{b} = \frac{(m+m')u}{m'u}, \quad \frac{c+d}{d} = \frac{(m+m')u_1}{m'u_1}, \quad \frac{e+f}{f} = \frac{(m+m')u_2}{m'u_2}.$$

Therefore we have

$$\frac{a+b}{b} = \frac{c+d}{d} = \frac{e+f}{f}. \qquad \text{Why?}$$

Algebraic Proof.

Given $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}.$

We may add 1 to each ratio without destroying the equality;

so we have $\dfrac{a}{b} + 1 = \dfrac{c}{d} + 1 = \dfrac{e}{f} + 1$ Why?

Adding: $\dfrac{a+b}{b} = \dfrac{c+d}{d} = \dfrac{e+f}{f}.$

Explain how your figure shows that adding 1 to the first set of ratios gives the second set of ratios.

Explain the changes that would need to be made in the proof in order to have this theorem read:

In a series of equal ratios, the sum of the first antecedent and consequent is to the first antecedent, as the sum of the second antecedent and consequent is to the second antecedent, and so on.

202. Theorem V. *In a series of equal ratios, the difference between the first antecedent and consequent is to the first consequent as the difference between the second antecedent and consequent is to the second consequent, as the difference between the third antecedent and consequent is to the third consequent.*

Show this true by measurement, then show it to be true by geometric proof. Algebraically it may be stated:

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then $\frac{a-b}{b} = \frac{c-d}{d} = \frac{e-f}{f}$.

Make the algebraic proof of this by subtracting 1 from each of the given ratios.

Re-word the proposition so that the word antecedent shall take the place of consequent as the second term of each ratio, and prove.

203. Theorem VI. *In a series of equal ratios, the sum of the first antecedent and consequent is to their difference, as the sum of the second antecedent and consequent is to their difference, and so on.*

Show by measurements of lines. Prove by geometry. Write in algebraic language and prove by algebra.

Suggestion. The algebraic proof can readily be made by dividing the algebraic expression of Theorem IV by the algebraic expression of Theorem V.

204. Definitions. The operation of passing from $\frac{a}{b} = \frac{c}{d}$

to $\frac{a+b}{b} = \frac{c+d}{d}$ is called "**composition**";

to $\frac{a-b}{b} = \frac{c-d}{d}$ is called "**division**";

to $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ is called "**composition and division.**"

The word division is here used in a sense entirely different from its usual meaning.

205. Theorem VII. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Given the equal ratios: $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots\cdot.$

To prove that $\frac{a + c + e \ldots\ldots}{b + d + f \ldots\ldots} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \cdots\cdot.$

Proof. Since the ratios are equal, let the common value be r.

Then $\frac{a}{b} = r, \quad \frac{c}{d} = r, \quad \frac{e}{f} = r, \cdots\cdot.$

Or $a = br, \quad c = dr, \quad e = fr, \cdots\cdot.$

Adding: $a + c + e + \ldots\ldots = (b + d + f + \ldots\ldots)\, r.$

Therefore $\frac{a + c + e + \ldots\ldots}{b + d + f + \ldots\ldots} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots\cdot.$

Exercise. Consider any number of equal fractions:

$$\frac{2}{3} = \frac{6}{9} = \frac{12}{18} = \frac{20}{30}.$$

Form a new fraction by adding together several or all the numerators and the corresponding denominators. Show that the new fraction is equal to $\frac{2}{3}$.

Apply this principle to show that if the fractions $\frac{5 + x}{1 + 2x}$ and $\frac{3 - x}{3 - 2x}$ are equal, then each of them must equal 2.

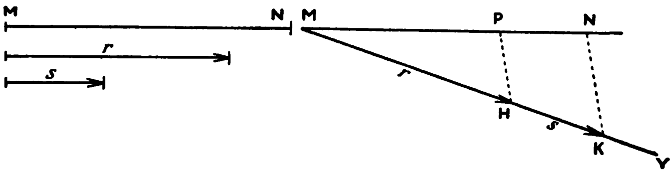
We can make use of Theorem I to divide a line-segment in a given ratio.

206. Definition. Given a line segment AB, which is divided into two parts AC and CB by a point C. The ratio of these segments is $AC : CB$, and line AB is said to be *divided internally* in the ratio $AC : CB$. Here point C falls between A and B.

Examples.

$$\frac{AC}{CB} = \frac{2}{3} \qquad \frac{AC}{CB} = \frac{5}{2} \qquad \frac{AC}{CB} = \frac{6}{3} = \frac{2}{1} = 2.$$

207. Problem I. *To divide a line-segment internally in a given ratio.*



Given the line-segment MN and the segments r and s.

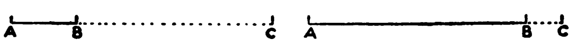
To divide the line-segment MN internally in the ratio of $r : s$.

Analysis. We can form a pencil of two rays, one being the line-segment MN and the other an indefinite straight line MY, M being the vertex of the pencil. On MY lay off MH equal to r, and add to it HK equal to s. Join N and K. Now by drawing a line through H parallel to NK, have the line-segment divided in the ratio of $r : s$.

Construct and prove that $MP : PN = r : s$.

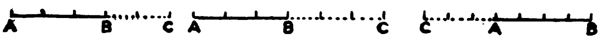
Discussion. Consider the case where the vertex of the pencil is taken at N.

208. Definition. Given a line-segment AB and a point C on AB *produced.* Then C is said to *divide AB externally* in the ratio $AC : CB$.



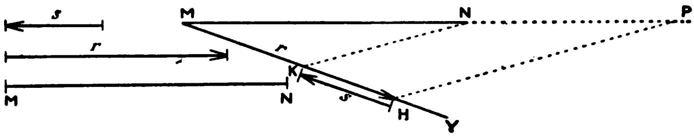
Notice that the order of the letters in the ratio is exactly the same as for internal division.

Examples.



$$\frac{AC}{CB}=\frac{5}{2} \qquad \frac{AC}{CB}=\frac{6}{3}=2 \qquad \frac{AC}{CB}=\frac{3}{7}.$$

209. Problem II. *To divide a given line-segment externally in a given ratio.*



Given line-segment MN and segments r and s.

To divide MN externally in the ratio r : s.

We follow the same plan as in Problem I, the only difference being that $HK = s$ is laid off from H toward M instead of from H toward Y. We now join K to N as before, and then through H draw a line parallel to KN. This will meet MN produced at P, which is the required point of external division.

Construct and prove that $\frac{MP}{PN}=\frac{MH}{HK}=\frac{r}{s}$.

Discussion. Consider the case where the vertex of the pencil is taken at N.

Those who have studied "First Course" will notice the correspondence between laying off segments r and s on MY and the addition of line segments as explained on pages 54–57. External division corresponds to dividing a line into two segments, one positive and the other negative. The given segment is divided into two parts one positive and the other negative.

To illustrate this, let the student divide a segment of 8 units in the ratio $5 : -2$; one of 12 units in the ratio $4 : -7$; one of $\sqrt{21}$ units in the ratio $10 : -3$.

210. Exercises.

1. By measurement determine the approximate ratio of division of the segments MN in Arts. **207** and **209.**

2. Draw a segment 10 units long; divide it internally in the ratio 3 : 5; externally in the ratio 3 : 5.

3. Draw a line-segment 10 units long, and a square 4 units on each side. Divide the segment internally and externally in the ratio of side : diagonal of the square.

If $a : b = c : d$, prove the following relations:

4. $\frac{a^2}{b^2} = \frac{c^2}{d^2}$.

5. $\frac{a^2}{ab} = \frac{c^2}{cd}$.

6. $\frac{a^2 + b^2}{b^2} = \frac{c^2 + d^2}{d^2}$.

7. $\frac{a^2 + b^2}{a^2 - b^2} = \frac{c^2 + d^2}{c^2 - d^2}$.

8. $2a : 3b = 2c : 3d$.

9. $2a + 3b : b = 2c + 3d : d$.

10. $\frac{ma}{mb} = \frac{nc}{nd}$.

11. $\frac{ma + nb}{nb} = \frac{mc + nd}{nd}$.

12. $\frac{4a + 5b}{4a - 5b} = \frac{4c + 5d}{4c - 5d}$.

13. $\frac{3a^2 + 5b^2}{b^2} = \frac{3c^2 + 5d^2}{d^2}$.

14. $\frac{\sqrt{a^2 + b^2}}{b} = \frac{\sqrt{c^2 + d^2}}{d}$.

15. $\frac{\sqrt{3a^2+5b^2}}{b} = \frac{\sqrt{3c^2+5d^2}}{d}$.

16. $a^3 : b^3 = c^3 : d^3$.

17. $a^n : b^n = c^n : d^n$.

18. $a^3 + b^3 : b^3 = c^3 + d^3 : d^3$.

19. $a^3 - b^3 : c^3 - d^3 = b^3 : d^3$.

20. $a^2 + ab + b^2 : c^2 + cd + d^2 = b^2 : d^2$.

21. $a^2 - ab + b^2 : c^2 - cd + d^2 = b^2 : d^2$.

22. $a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$.

23. $\sqrt{a^2 + ab + b^2} : \sqrt{c^2 + cd + d^2} = \sqrt[3]{a^3 - b^3} : \sqrt[3]{c^3 - d^3}$. (See Exercises 19, 20.)

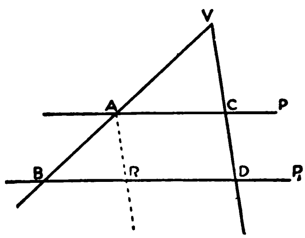
24. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that $\frac{3a + 2c + 5e}{3b + 2d + 5f} = \frac{a}{b}$.

25. If $\sqrt{2} - x : x = x - 3 : \sqrt{2} + x$, find the value of x and check.

26. If $\sqrt{x-2} - \sqrt{x+2} : \sqrt{x+3} = \sqrt{x-3} : \sqrt{x-2} + \sqrt{x+2}$, find the value of x and check.

Going back to our figure of the pencil of lines cut by parallels, there are still equal ratios which we have not investigated.

211. Theorem VIII. *If a pencil of lines is cut by a pencil of parallels, the segments of the lines of the pencil (measured from the vertex) are proportional to the corresponding segments of the parallels.*



Given the pencil of lines with vertex V. VA and VB are the segments cut on one of the rays by the parallels P and P_1. AC and BD are the corresponding segments cut on the parallels.

To prove that $\frac{VA}{VB} = \frac{AC}{BD}$.

Analysis. Since the only theorem that we have had on this subject deals with ratios between segments of the rays of the pencil of lines, we shall form a pencil such that VA, VB, AC, BD shall be the segments of its rays, or equal to segments of its rays. If we draw a line through A parallel to VD, we shall have such a pencil. Point B is the vertex of this pencil. AC is not a segment of a ray of the pencil, but if we can show AC equal to RD, R being the point where the line drawn through A cuts BD, we shall have proved our theorem.

Proof. Give this proof in full.

212. Theorem IX. *If a pencil of lines is cut by a pencil of parallels, the corresponding segments cut on the parallels are in proportion.*

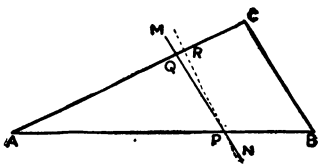
Draw figure, analyze, and prove by application of Theorem VIII.

By limiting Theorem I to two rays and two parallels we have:

213. Theorem X. *If a line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.*

The converse of this theorem is:

214. Theorem XI. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Given the triangle ABC with the line MN cutting the side AB at P and side CA at Q, forming the proportion $\frac{AP}{PB} = \frac{AQ}{QC}$.

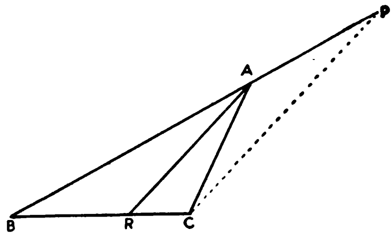
To prove that line MN is parallel to side BC.

Analysis. We shall follow the usual plan in the proof of a converse theorem, that is, we shall draw through point P a line PR parallel to side BC, and prove that PR coincides with MN. This will prove our theorem.

Proof. Prove this theorem by answering the following questions. Since PR was drawn parallel to BC, what proportion exists? Why? What proportion exists from the assumption of the theorem? Why? What two ratios are equal to one another, both being equal to the same ratio? Can this be possible without R and Q coinciding?

State theorem proved.

215. Theorem XII. *If the interior angle of a triangle is bisected, and the bisector extended until it cuts the opposite side, it divides that side into segments proportional to the other two sides.*



Given the triangle ABC with the angle A bisected by the line AR, forming the segments BR and RC.

To prove that $\dfrac{BR}{RC} = \dfrac{AB}{AC}$.

Analysis. We can prove this theorem by forming a pencil of lines cut by a pencil of parallels. This pencil must have BR and RC for segments of one of its rays. BA can be a segment on another ray. This suggests immediately that RA be one of the parallel lines and that we must draw another line parallel to RA through point C. Extending BA through A until it meets this line in point P, we have a proportion which contains BR, RC, and BA, AP. These are all of the terms in the proportion that we are trying to prove excepting CA, which must take the place of AP. So that if our proportion is true at all, we must be able to prove that the line CA is equal to the line AP.

We can show that CA equals AP if we can show that angle APC equals angle PCA. We can show angle APC equal to angle PCA, if we can show these equal respectively to the angles BAR and RAC, which themselves were made equal.

Give this proof complete.

Exercise. If $AB = AC$, what fact about an isosceles triangle is brought out by this theorem?

216. Theorem XIII. *If the exterior angle of a triangle is bisected, the bisector divides the opposite side externally into segments proportional to the other two sides.*

Analyze and prove on the same plan that you proved Theorem XII.

Notice that the bisector of the interior angle of a triangle divides the opposite side in the same ratio as the bisector of the exterior angle at the same vertex. That is the external division of the line is in the same ratio as the internal division of the line.

Definition. When a line is divided internally and externally in the same ratio, the line is said to be divided **harmonically.**

217. Exercises.

1. In Theorem I show that the rectangle of a and d equals the rectangle of b and c. State this in the form of a theorem.

2. Show that $a = \frac{bc}{d}$; $b = \frac{ad}{c}$. (In Theorem I.)

3. Construct the line whose length is equal to $\frac{7 \cdot 2}{9}$.

4. In Theorem I if $a = r$, $b = 4r - 1$, $c = 3$, and $d = 12r$, find the value of r.

5. In Theorem I if $a = m$, $b = 2m + 4$, $c = 6m - 5$, $d = 13m + 5$, find the value of m.

6. Two parallel lines cut a pencil of two rays, cutting the segments a and b on one ray, and c and d on the other. If segment b is 4 units more than twice segment a, segment c is 5 units less than 6 times segment a, and segment d is 5 units more than 13 times segment a, find the lengths of segments a, b, c and d.

Note. In solving this exercise notice that it leads to the algebraic expressions in Exercise 5.

7. In Theorem I suppose $a = p$, $b = \frac{1}{6}p - 1$, $c = 3p$, and $d = \frac{p-1}{2}$, write a problem leading to these expressions and solve.

8. The sides of a triangle are 7, 8, and 12. Find the length of the segments into which the bisector of each angle divides the opposite side. Use both the exterior and interior angles. Construct the figure and check results by measurements.

9. In a given triangle one side is $5\frac{1}{3}$ units. The angle opposite this side is bisected and the bisector divides this side into two segments such that if 4 units be added to the shorter it is equal to one of the sides of the triangle, and if 4 units be added to three times the shorter segment, it is equal to the other side. Find the length of the sides of the triangle and the length of the segments into which the first side is divided.

10. The diagonals of a trapezoid divide each other into parts proportionate to the bases of the trapezoid.

Suggestion. Regard the point of intersection of the diagonals as the vertex of a pencil of two rays.

11. If a trapezoid has one base twice as long as the other, into what ratio will the diagonals divide each other?

12. The medians of a triangle meet in a point which is two-thirds of the distance from any vertex to the mid-point of the opposite side.

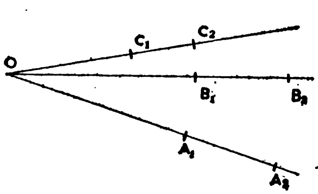
Suggestion. Show that this is an application of Exercise 11 by noticing that the line joining the ends of two medians is parallel to the third side of the triangle and half as long.

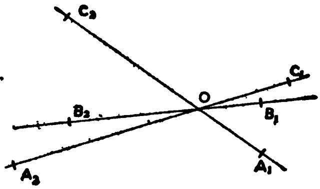
13. The line joining the mid-points of the bases of a trapezoid divides each of the other sides externally into parts proportional to the bases.

14. A line segment 10 units long is divided internally in the ratio 2:3. Determine by construction an external point of division so that the line shall be divided harmonically. Is there only one such point?

PART II—SIMILAR FIGURES

218. Definition. Draw a pencil of lines O. On each ray mark a point. Call these points A_1, B_1, C_1 .... On the rays mark another set of points A_2, B_2, C_2 .... in such a way that $\frac{OA_1}{OA_2} = \frac{OB_1}{OB_2} = \frac{OC_1}{OC_2} \cdots$.

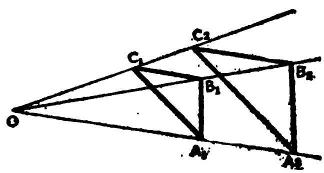


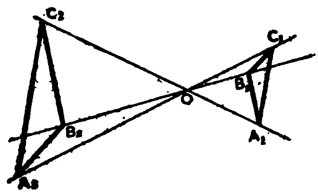


We call these sets of points, A_1, B_1, C_1 .... and A_2, B_2, C_2 ...., **similar sets of points,** or **similar systems of points.**

If we join these similar systems of points by straight lines, the figures formed are called **similar polygons.**

Point O is called the **center of similitude.** The ratio of the two segments on any ray through this point is constant for all the rays.

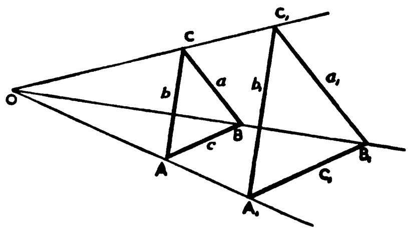




Notation. The symbol for similar is $\sim$, an *s* lying on its side.

In either of the two preceding figures we have $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$, by the definition of similar figures. It then follows directly from Theorems X and XI, page 181, that homologous sides of these triangles are proportional and that they are parallel, and therefore that the triangles are mutually equiangular. (See Corollaries 1 and 2 below.) This result is often used as the starting point in the definition of similar triangles, that is, two triangles are *defined* to be similar when their homologous sides are proportional and their homologous angles are equal.

Corollary 1. *If two triangles are similar, their corresponding sides are proportional.*



Suggestion. Prove this by showing that

$$\frac{a}{a_1} = \frac{OC}{OC_1} = \frac{b}{b_1} = \frac{OA}{OA_1} = \frac{c}{c_1}.$$

Corollary 2. *If two triangles are similar, they are mutually equiangular.*

Suggestion. Use Art. **214** and Art. **62.**

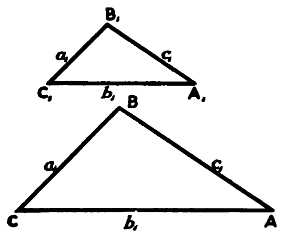
Combining corollaries **1** and **2**, we have: *If two triangles are similar, they are mutually equiangular and their corresponding sides are proportional.*

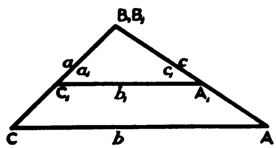
219. The theorems on similar triangles are very closely related to those on congruent triangles. In fact, as will be seen, the congruent triangles are but special cases of similar triangles. This similarity begins in the treatment of the two subjects.

In beginning the discussion of congruent triangles, we first defined them as triangles which when placed one upon the other could be made to coincide. Then we proved that if triangles had certain characteristics, we could always, if we chose, place them together and make them coincide. Then, if we could show triangles to have these characteristics, we should know them to be congruent without placing them together.

So in the treatment of similar figures, our definition assumes them so arranged that their corresponding points lie on the rays of a pencil, whose vertex divides these rays in a series of equal ratios. The figures are then said to be in *similar perspective.* We shall prove that if triangles have certain characteristics, they can always be placed in similar perspective, and hence it will not be necessary to so place them in order to know that they are similar. We shall know them to be similar by these characteristics.

220. Theorem XIV. *Two triangles are similar if they have two angles of the one equal respectively to two angles of the other.*





Given the triangle ABC and $A_1B_1C_1$ with the angle A equal to angle A_1, and angle B equal to angle B_1.

To prove that triangles ABC and $A_1B_1C_1$ are similar.

Analysis. We can prove that $\triangle$ ABC and $A_1B_1C_1$ are similar if we can prove that their sides are proportional, that is, prove that $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$. We can show that $\frac{a}{a_1} = \frac{c}{c_1}$ if, when we place the two triangles together with the equal angles B and B_1 coinciding, we can prove that side b is parallel to side b_1. We can show that side b is parallel to b_1 by showing that they are cut by the transversal AB, and that the corresponding angles BAC and $B_1A_1C_1$ are equal. We can show that $\frac{b}{b_1} = \frac{c}{c_1}$ by again placing the triangles together with the equal angles A and A_1 coinciding and showing that the sides a and a_1 are parallel.

Proof. Place $\triangle$ ABC and $A_1B_1C_1$ together so that $\angle$ B will coincide with $\angle$ B_1. (Why will they coincide?)

Then point A_1 will fall on side c or on side c produced, and point C_1 will fall on side a or on side a produced.

$$\angle BAC = \angle B_1A_1C_1. \qquad \text{Why?}$$

$$\therefore \quad b \parallel b_1. \qquad \text{Why?}$$

$$\therefore \quad \frac{b}{b_1} = \frac{c}{c_1}. \qquad \text{Why?}$$

In like manner, placing the triangles together with the equal angles A and A_1 coinciding, prove that $\frac{a}{a_1} = \frac{c}{c_1}$.

$$\therefore \quad \frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}.$$

$$\therefore \quad \triangle ABC \sim \triangle A_1B_1C_1. \qquad \text{Why?}$$

State the theorem proved.

Corollary 1. *Two triangles are similar if they have their sides parallel each to each, or their sides perpendicular each to each.*

Corollary 2. *Two equilateral triangles are similar.*

Corollary 3. *Two right triangles are similar if they have an acute angle of the one equal to an acute angle of the other.*

Exercises.

1. Construct a triangle similar to a given triangle, using the above theorem.

2. Construct a triangle similar to a given triangle, using each of the above corollaries.

3. The perpendicular from the vertex of the right angle of a right triangle to the hypotenuse forms two right triangles similar to the first triangle.

4. The sides of a triangle are each 4 times as long as the sides of another triangle. What can you say about their angles?

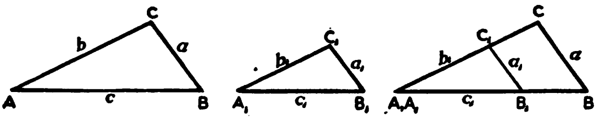
5. If $\triangle ABC \sim \triangle A_1B_1C_1$, and if $2\,AB = 3\,A_1B_1$, what is the ratio of the perimeters? Construct two such triangles.

6. Construct a triangle with sides 3, 5, 6 units long respectively, and prolong each side in both directions. By drawing in three more lines construct three triangles with sides twice as long as those of the first triangle and similar to it and to each other.

221. Theorem XV. *The corresponding altitudes of similar triangles are proportional to their corresponding sides.*

Suggestion. Apply Art. **220,** Cor. 3.

222. Theorem XVI. *If two triangles have an angle of the one equal to an angle of the other, and the including sides proportional, the triangles are similar.*



Given the triangles ABC and $A_1B_1C_1$ with angle A equal to angle A_1 and the including sides of such length that $\frac{b}{b_1} = \frac{c}{c_1}$.

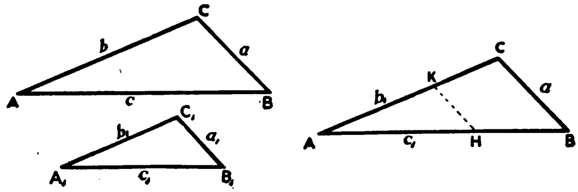
To prove that triangles ABC and $A_1B_1C_1$ are similar.

Analysis. We can prove this if we can prove two angles of one equal respectively to two angles of the other. Since $\angle A$ equals $\angle A_1$, we have but to prove that $\angle B$ equals $\angle B_1$. By placing the two triangles together with equal angles A and A_1 coinciding, we can prove that $\angle s$ B and B_1 are equal if we can prove that side a is parallel to side a_1. This we can prove by showing that side a_1 divides the sides b and c proportionally.

Proof. Give a complete proof of this theorem.

Exercise. Construct a triangle similar to a given triangle, using the above theorem.

223. Theorem XVII. *If two triangles have their sides proportional they are similar.*



Given the triangles ABC **and** $A_1B_1C_1$ **with their sides of such length that** $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$.

To prove that triangles ABC and $A_1B_1C_1$ ***are similar.***

Analysis. Since we have no angles given in this proposition, we shall not be able to place the triangles together, and know that their sides will take the same direction. We shall make a pencil with vertex A. On ray AC we lay off AK equal to b_1. On ray AB we lay off AH equal to c_1. Join H and K. We can now prove our proposition if we can prove $\triangle\triangle\ AHK$ and ABC similar, and then prove that $\triangle\ AHK$ is congruent to $\triangle\ A_1B_1C_1$.

Proof. $HK \parallel BC$. (See Theorem XI.) Why?

$\therefore$ $\triangle\ AHK \backsim \triangle\ ABC$. Why?

In $\triangle\triangle\ AHK$ and $A_1B_1C_1$,

$AK = b_1$,

$AH = c_1$, by construction.

Since $\triangle\ AHK \backsim \triangle\ ABC$,

$\therefore$ $\frac{c}{c_1} = \frac{a}{HK}$. Why?

But $\frac{c}{c_1} = \frac{a}{a_1}$. Why?

$\therefore$ $\frac{a}{HK} = \frac{a}{a_1}$. Why?

$\therefore$ $HK = a_1$. Why?

$\therefore \qquad \triangle AHK \cong \triangle A_1B_1C_1.$

$\therefore \qquad \triangle A_1B_1C_1 \sim \triangle ABC.$ Why?

224. Exercises.

1. Construct a triangle similar to a given triangle but with half the perimeter.

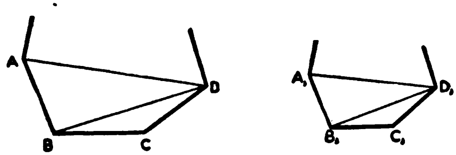
2. Construct a triangle and draw the lines joining the midpoints of its sides. Show that the four triangles so formed are similar to the given triangle and congruent to each other.

3. Construct a triangle similar to a given triangle and having its perimeter equal to a given straight line.

Suggestion. In the figure of Art. **87**, make AB equal to the given perimeter; then lay off on AX segments equal to the sides of the given triangle instead of equal segments.

State theorems on congruent triangles that correspond to theorems on similar triangles, and explain how the former are special cases of the latter.

225. Theorem XVIII. *If two polygons are similar they can be divided into the same number of similar triangles, similarly placed.*



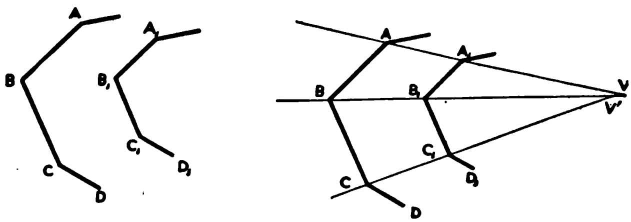
Suggestion. Since the polygons are similar they can be placed in similar perspective. Then the triangles are similar by definition.

Corollary 1. *If two polygons are similar, they are mutually equiangular, and their corresponding sides are proportional.*

Corollary 2. *The perimeters of similar polygons have the same ratio as any two corresponding sides.*

Suggestion. State the second part of Corollary 1 in the form of a series of equal ratios and apply Theorem VII.

226. Theorem XIX. *If two polygons are mutually equiangular, and have their corresponding sides proportional, they are similar.*



Given the two polygons $ABCD \cdots\cdots$ **and** $A_1B_1C_1D_1 \cdots\cdots$ **with sides of such length that** $\frac{AB}{A_1B_1} = \frac{BC}{B_1C_1} = \frac{CD}{C_1D_1} = \frac{DE}{D_1E_1} = \cdots\cdots$**, and with angles such that** $\angle A = \angle A_1$, $\angle B = \angle B_1$, $\angle C = \angle C_1, \cdots\cdots$.

To prove that polygon ABCD $\cdots\cdots$ ***is similar to polygon*** $A_1B_1C_1D_1 \cdots\cdots$.

Analysis. We can show these two polygons similar if we can show that they can be so placed that the joins of their corresponding points will form a pencil of lines. We start by placing the side AB parallel to side A_1B_1, then the remaining corresponding sides will fall parallel to one another respectively. The joins of A, A_1 and B, B_1 will meet in a point. Why? Let this point be V. Now if we can prove that join of C, C_1 will meet BB_1 in the same point, we will have proved our proposition. Let V' be the point in which CC_1 meets BB_1. We can prove that V and V' coincide if we can prove that $\frac{BV}{B_1V} = \frac{BV'}{B_1V'}$ by proving them each equal to $\frac{BC}{B_1C_1}$.

Give this proof complete.

In the figure above it is understood of course that the lines there shown represent only portions of the entire polygons, which may have any number of sides.

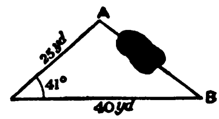
227. Exercises.

1. The sides of a triangle are 5, 7, and 10. Construct a similar triangle whose side corresponding to side 5 is 11.

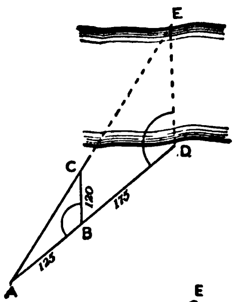
2. On cross section paper draw a map of a state, with irregular boundary lines straightened out, keeping the area about the same.

3. Take measurements of the ground outline of your school building. Reproduce it in correct proportions on cross section paper.

4. Suppose that in measuring from one point to another, an obstacle is in your way so that you cannot go directly. Show that by taking measurements as in the figure, and drawing to a suitable scale, you can find the length desired by measurement of the figure.

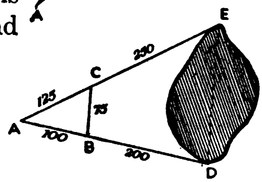


Make such measurements.

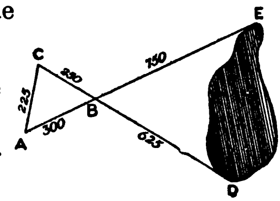


5. Show how, by measuring the sides of a triangular piece of ground, you could determine the angles by construction and measurement.

6. In the adjacent figure line BC is drawn so that $\angle ABC = \angle ADE$. Find the width of the river. Check by drawing a figure to scale.



7. In the adjacent figure prove that BC is parallel to DE. Calculate the length of DE. Check by drawing the figure to scale.



8. In the adjoining figure prove that AC is parallel to DE. Calculate the length of DE. Check by drawing the figure to scale.

9. Let the student go out and take measurements, for the purpose of computing a distance that cannot be measured.

10. A boy found that by holding a pencil and a foot-rule at different distances from his eye but parallel to each other, he could make the length of the pencil just cover the length of the ruler. In this position the distance from his eye to one end of the pencil was 4 in. more than the length of the pencil, while the distance to the corresponding end of the ruler was 2 in. more than twice the length of the pencil. Find the length of the pencil.

11. A triangular flower bed is surrounded by a uniform walk; the entire plot is fenced in. The length of the fence along the shortest side is 12 ft. The length of the shortest side of the bed is one foot less than half the next side and the fencing along that side is 9 ft. longer than the side. If the entire length of fencing is $61\frac{1}{2}$ ft. find the length of each side of the flower bed.

12. Write a problem involving similar triangles which will lead to a quadratic equation and solve. (See Exercise 11.)

13. Write a problem involving Theorem XII, which will lead to a quadratic equation. Solve.

14. Write a problem involving Theorem X which will lead to a quadratic equation. Solve.

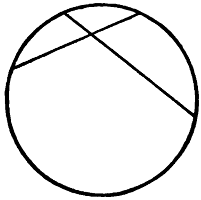
15. The altitudes of a triangle divide each other proportionally. In the proof use triangles containing both acute and obtuse angles.

16. Draw a circle, two intersecting chords in the circle, and chords joining the ends of the intersecting chords. Name and prove similar triangles formed. See Art. **228,** p. 195.

17. Draw a circle with any radius, say 2 inches, and in it draw a chord 3 inches long. Through a point 1 inch from the end of this chord draw another chord which shall be divided at this point in the ratio 3:1.

Suggestion. First calculate the segments of the second chord by use of Exercise 16, then construct them.

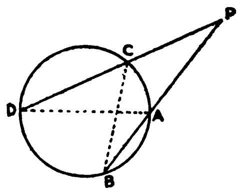
228. Theorem XX. *If two intersecting chords are drawn in a circle, the product of the segments of one chord is equal to the product of the segments of the other.*



Suggestion. Draw chords to join the ends of the intersecting chords and prove triangles similar as in Exercise 16. Form a proportion, and clear the equation of fractions.

This proposition is frequently stated:

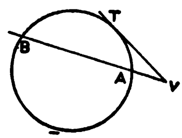
If a pencil of lines is cut by a circumference, the product of the two segments cut from the vertex of the pencil is constant no matter which line is taken.



Prove that this statement is true when the vertex of the pencil is without the circumference.

Prove triangles DBP and APC similar.

By turning one of the rays gradually about the vertex of your pencil show that:



Corollary. *The tangent from the vertex of a pencil of lines is a mean proportional between the segments of any other ray of the pencil.*

229. Exercises.

1. The square of half the shortest chord that can be drawn through a point within a circle is equal to the product of the segments of the diameter through the point.

This may be stated:

One-half of the shortest chord drawn through a point within a circle is a mean proportional between the segments of the longest chord through that point.

Another statement is:

The perpendicular, drawn from the vertex of the right angle of a right triangle to the hypotenuse, is a mean proportional between the segments of the hypotenuse.

2. Explain how the above exercise may be used to construct a mean proportional between two given lines.

3. How far is a point from a circular lake 7 miles in diameter, if the distance from the point to the shore measured along the tangent line is 6 miles less than twice the distance measured along a line which, if continued, would pass through the center of the lake.

Solve Exercise 3 without using the Pythagorean theorem.

4. Write a problem involving Theorem XX, which will lead to a quadratic equation. Solve.

5. If two circles are tangent and three secants are drawn through the point of tangency, prove that the chords joining the ends of these secants form similar triangles.

6. At the ends of a diameter of a circle, with center C and radius r, erect perpendiculars to it. Draw a line tangent at any point P and cutting the ⊥s at A and B. Prove △ ABC is right-angled and that $AP \cdot PB = r^2$.

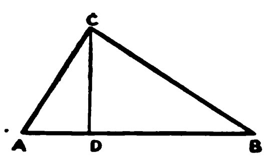
7. Turn to the figure of Ex. 24, Art. **168**, and name all of the similar triangles that you can find, and all of the proportions, giving reasons.

230. Theorem XXI. *If a perpendicular is dropped from the vertex of the right angle of a right triangle to the hypotenuse:*

(*a*) *Two triangles will be formed which are similar to the original triangle and similar to each other;*

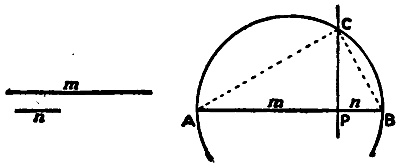
(*b*) *Either side is a mean proportional between the hypotenuse and the adjacent segment;*

(*c*) *The perpendicular is a mean proportional between the segments of the hypotenuse.*



Analyze, and prove.

Problem V. *To construct a mean proportional between two given lines.*



Given the lines m **and** n.

To construct a line which is a mean proportional between m and n.

Analysis. Suppose the figure drawn. We see that we shall have PC a mean proportional between m and n, if we can have angle C a right angle and AB the hypotenuse of a right triangle, AB being the sum of m and n, and P being the point which separates these two segments. That is we are required to locate a point C, from the conditions given, first that C must lie on the perpendicular drawn to AB at point P, second that C must be the vertex of a right angle of a triangle of which AB is the hypotenuse.

The locus from the first condition is the perpendicular. The locus to fulfil the second condition is the circumference on AB as a diameter. The intersection of these two lines is the required point C.

Make this construction and prove.

Corollary. *To construct a square equal to a given rectangle.*

231. Exercises.

1. Construct a square which equals half a given rectangle.
2. Construct a square which equals twice the square constructed in Exercise 1. How does this square compare with the given rectangle of Exercise 1?
3. Construct a square which is one-fifth of a given square.
4. The diagonal of one square is three times the diagonal of a second square. Find the ratio of their areas.

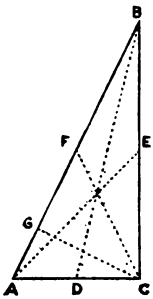
5. The area of a square is 10 square units. Find the length of the line drawn from one corner to the middle of one of the opposite sides.

6. A line CD of varying length, perpendicular to AB, moves with its end C along AB so that $\overline{CD}^2 = AC \cdot CB$. Find the locus of point D.

7. In a right triangle ABC, $\angle A = 45°$. If $\angle A$ is bisected, and the bisector cuts side BC at D, find the ratio of BD to DC.

8. If a circle is inscribed in a right triangle, the sum of diameter and hypotenuse equals the sum of the other two sides.

9. In right triangle ABC, let $AB = c$, $AC = b$, $CB = a$; CG is $\perp AB$, and AE, BD, CF are medians.



Show that $CG = \frac{ab}{c}$; $AG = \frac{b^2}{c}$; $GB = \frac{a^2}{c}$;

$AE = \frac{1}{2}\sqrt{4b^2 + a^2}$;

$BD = \frac{1}{2}\sqrt{4a^2+b^2}$; $CF = \frac{1}{2}c$;

radius of circumscribed $\odot = \frac{1}{2}c$;

radius of inscribed $\odot = \frac{a+b-c}{2}$.

10. Draw an accurate figure like that in Exercise 9, taking $a = 2$ in., and $b = 4$ in. Calculate the lengths of all the lines, and check by measurement.

11. If r and r' are the radii of two concentric circles, respectively, find the length of a chord of the outer circle that is tangent to the inner circle. First use the Pythagorean theorem in the solution, then solve again by use of Theorem XX, page 195. To apply this theorem draw one of the chords mentioned, and through the point where it touches the inner circle draw a diameter of the outer circle perpendicular to the chord.

In the following find the mean proportional by drawing and by Algebra. Compare results, by counting if rational, by constructing by the Pythagorean theorem if irrational.

12. 9 and 225. **13.** 8 and 72. **14.** 34 and 136.

15. 6 and 3. **16.** $\sqrt{5}$ and $\sqrt{15}$. **17.** $\sqrt{21}$ and $\sqrt{14}$.

Without drawing find the mean proportional between

18. a and b. **19.** a^2 and b^2.

20. $\dfrac{a^2 - b^2}{a^2 - ab + b^2}$ and $\dfrac{a^3 + b^3}{a - b}$.

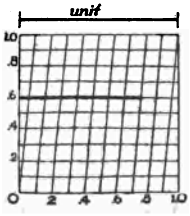
21. A railroad forms a chord across a circular lake whose diameter is d. To go to the half-way station one must go along the railroad a feet more than n times as far as to go by water from a point on the shore directly opposite the station. How far is it to the station by each route.

22. Write Exercise 21 with special values for the general numbers, and use the answers of that exercise as formulæ to solve your exercise.

23. Write a general problem involving Theorem XXI, leading to a quadratic equation. Solve.

24. Rewrite Exercise 23, giving special values to the general numbers used, and solve by using the answers to Exercise 23 as formulas.

232. Up to this time we have been obtaining the lengths of lines representing surd numbers by means of the Pythagorean theorem—but, as has been remarked, this is not the practical method. Approximate values as close as desired are used in practice. For the purpose of close measurement various instruments have been devised. A very simple one which you can make and use in the future is made as follows:



Line in a square of 10 units on your cross-section paper. Numbering the division points each way 0, 1, 2, ..., join the tenth point at the top to the ninth point at the bottom, and so on. Rule your own cross-section paper if none is at hand.

Regard the length of your square as 1 unit, and explain how your *diagonal scale* may be used to measure any number of tenths or hundredths of a unit. Give reason.

Make a diagonal scale by which any number of tenths or hundredths of an inch may be measured.

Explain how your compass may be used to take a required length stated in units, tenths, and hundredths, from the diagonal scale and lay it off on a straight line. If greater accuracy is required, dividers instead of the compass should be used.

PART III—TRIGONOMETRIC FUNCTIONS

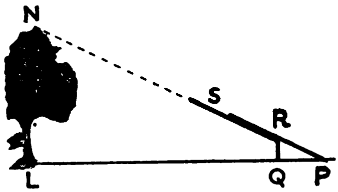
233. In the history of mathematics we read that Thales measured the heights of the Pyramids in Egypt. He did it in this way: At the time of day when he observed that the length of the shadow of a stick set perpendicularly to the ground was as long as the stick, he measured the length of the shadow of the pyramid. Did he work on correct mathematical principles?

Using the same plan that Thales used, take measurements and find the height of a smoke stack or some building, a telephone pole or electric light pole.

It will not be necessary for you to wait until the shadow is the same length as the stick that you use. You may take the measurement at any convenient time. Why? How does Thales' method show that he was limited in his knowledge of the subject of proportion?

The plan that Thales used required that the sun be shining, so as to produce shadows. This plan would not work in a forest.

A man measured the heights of trees with a home-made instrument of the following sort. He had two strips of wood hinged together at the ends. Running the length of one of these strips was a groove in which another strip worked smoothly. When he was ready to use this instrument, he would push one of the hinged strips, which had a sharp end, into the ground and turn the other strip and sight along it until it pointed to the top of the tree. He would then allow the strip in the groove to slide down until it touched the ground, which point he marked. With a steel tape he measured from this point to the foot of the stake and on to the foot of the tree. From the data he computed the height of the tree.



On what principle did he work?

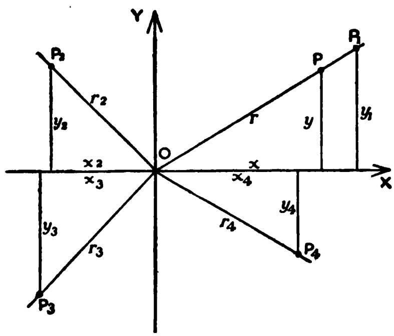
Exercise. If RQ is 5 feet, PQ is 10 feet, PL is 225 feet, how high is the tree?

Find the height of the tree if $PQ = a$, $PL = b$, and $QR = c$.

This method depends on the use of similar right triangles, and is much used in measurements of heights and distance.

234. In each of the methods described above it was necessary to use two triangles in the solution. We shall now investigate methods by which we can compute heights and distances from one triangle.

Draw two lines of reference at right angles to one another. These divide the plane of your paper into four quarters or quadrants. Art. **184**. Select point P any point in the plane. Four cases arise according to the quadrant in which you select point P. These are shown in the figure.



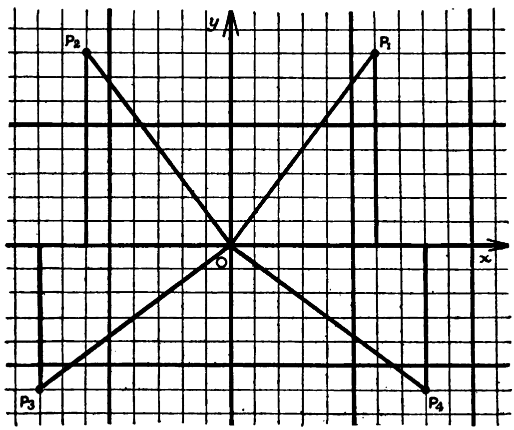
With reference to any one of these points, three lines are to be noted, namely, the **abscissa,** the **ordinate,** and the **distance** from O. For the point P in general these are indicated by the letters x, y, r respectively. For point P_1 we use the letters x_1, y_1, r_1, and so on for other points.

The student should carefully notice the signs of these quantities. Thus in the above figure x_1 stands for a positive number, x_2 for a negative number. Also y_1 and y_2 stand for positive numbers, and y_3 and y_4 stand for negative numbers. The distances r_1, r_2, r_3, r_4 are all considered positive.

Among these three quantities there are six possible ratios. Thus for x, y, r we have the ratios:

$$\frac{y}{r},\quad \frac{x}{r},\quad \frac{y}{x},\quad \frac{r}{y},\quad \frac{r}{x},\quad \frac{x}{y}.$$

The last three ratios are merely the reciprocals of the first three.



From the above figure, measure the lengths of x, y, r for each of the points, give each length its proper sign and then form the ratios; arrange your work in a table, as in Art. **235** below.

Thus for the point P_2, we find

$$x = -6, \qquad y = +8, \qquad r = 10.$$

Hence, $\frac{y}{r} = +0.80$, $\frac{x}{r} = -0.60$, $\frac{y}{x} = -1.33$, and so on.

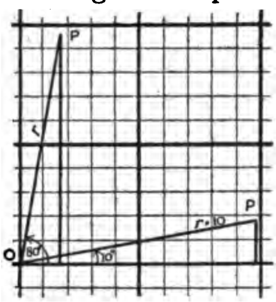
The question now arises, on what do these ratios depend?

Draw a careful figure showing a point P in the first quadrant, as P_1 in the figure above.

Measure x, y, r and calculate the ratio to the hundredths place.

Extend line OP_1 and on it mark a new point P'. Call the new values x', y', r'. Measure and calculate as before.

Are the ratios the same as before? Give a geometric proof that they must be so.



235. Then we discover this truth, that so long as the angle XOP remains the same, the ratios remain the same, even though the length of x, y, r are changed.

Allow r to remain the same and turn it about the origin until it makes an angle of 10° with the axis of the abscissa. This fixes the length of x and y. Take measurements and find the six ratios to hundredths place. Place in the form of a table, thus:

Angle	x	y	r	$\frac{y}{r}$	$\frac{x}{r}$	$\frac{y}{x}$	$\frac{r}{y}$	$\frac{r}{x}$	$\frac{x}{y}$

Turn r about the origin until it makes an angle of 80° with the axis of the abscissa. Take measurements and find the ratios as before. Does a change in the angle cause a change in the ratios?

Repeat the above for the angles 20°; 70°; 30°; 60°; 40°; 50°; 45°.

Do you notice any relation between any of these ratios? Prove that this relation should exist.

In this work what is the independent variable? What are the dependent variables? May we then say that these ratios are functions of the angle?

Mathematicians have given names to these ratios.

$\frac{y}{r}$ is called the **sine** of the angle.

$\frac{x}{r}$ is called the **cosine** of the angle.

$\frac{y}{x}$ is called the **tangent** of the angle.

$\frac{x}{y}$ is called the **cotangent** of the angle.

$\frac{r}{x}$ is called the **secant** of the angle.

$\frac{r}{y}$ is called the **cosecant** of the angle.

If we let ϕ (Greek letter *Phi*) be the number of degrees in the angle, these functions are usually written:

$$\sin\phi = \frac{y}{r}, \qquad \cot\phi = \frac{x}{y},$$

$$\cos\phi = \frac{x}{r}, \qquad \sec\phi = \frac{r}{x},$$

$$\tan\phi = \frac{y}{x}, \qquad \csc\phi = \frac{r}{y}.$$

Note. The student should have a care, in speaking or writing these functions of the angle, not to drop the angle. There is no such a thing as a sine in itself. It is always the sine of an angle. So with the other functions.

Compare your table with the one found on following page.

236. Table of Trigonometric Functions.

	sin φ	cos φ	tan φ	cot φ	sec φ	cosec φ	
0°	0.000	1.000	0.000	∞	1.000	∞	90°
1	0.017	0.999	0.017	57.290	1.000	57.299	89
2	0.035	0.999	0.035	28.636	1.001	28.654	88
3	0.052	0.999	0.052	19.081	1.001	19.107	87
4	0.070	0.998	0.070	14.301	1.002	14.335	86
5°	0.087	0.996	0.087	11.430	1.004	11.474	85°
6	0.104	0.995	0.105	9.514	1.006	9.567	84
7	0.122	0.993	0.123	8.144	1.008	8.206	83
8	0.139	0.990	0.140	7.115	1.010	7.185	82
9	0.156	0.988	0.158	6.314	1.012	6.393	81
10°	0.174	0.985	0.176	5.671	1.015	5.759	80°
11	0.191	0.982	0.194	5.145	1.019	5.241	79
12	0.208	0.978	0.213	4.705	1.022	4.810	78
13	0.225	0.974	0.231	4.332	1.026	4.445	77
14	0.242	0.970	0.249	4.011	1.031	4.134	76
15°	0.259	0.966	0.268	3.732	1.035	3.864	75°
16	0.276	0.961	0.287	3.487	1.040	3.628	74
17	0.292	0.956	0.306	3.271	1.046	3.420	73
18	0.309	0.951	0.325	3.078	1.051	3.236	72
19	0.326	0.946	0.344	2.904	1.058	3.072	71
20°	0.342	0.940	0.364	2.748	1.064	2.924	70°
21	0.358	0.934	0.384	2.605	1.071	2.790	69
22	0.375	0.927	0.404	2.475	1.079	2.670	68
23	0.391	0.921	0.424	2.356	1.086	2.559	67
24	0.407	0.914	0.445	2.246	1.095	2.459	66
25°	0.423	0.906	0.466	2.144	1.103	2.366	65°
26	0.438	0.899	0.488	2.050	1.113	2.281	64
27	0.454	0.891	0.510	1.963	1.122	2.203	63
28	0.469	0.883	0.532	1.881	1.133	2.130	62
29	0.485	0.875	0.554	1.804	1.143	2.063	61
30°	0.500	0.866	0.577	1.732	1.155	2.000	60°
31	0.515	0.857	0.601	1.664	1.167	1.942	59
32	0.530	0.848	0.625	1.600	1.179	1.887	58
33	0.545	0.839	0.649	1.540	1.192	1.836	57
34	0.559	0.829	0.675	1.483	1.206	1.788	56
35°	0.574	0.819	0.700	1.428	1.221	1.743	55°
36	0.588	0.809	0.727	1.376	1.236	1.701	54
37	0.602	0.799	0.754	1.327	1.252	1.662	53
38	0.616	0.788	0.781	1.280	1.269	1.624	52
39	0.629	0.777	0.810	1.235	1.287	1.589	51
40°	0.643	0.766	0.839	1.192	1.305	1.556	50°
41	0.656	0.755	0.869	1.150	1.325	1.524	49
42	0.669	0.743	0.900	1.111	1.346	1.494	48
43	0.682	0.731	0.933	1.072	1.367	1.466	47
44	0.695	0.719	0.966	1.036	1.390	1.440	46
45°	0.707	0.707	1.000	1.000	1.414	1.414	45°
	cos φ	sin φ	cot φ	tan φ	cosec φ	sec φ	

237. Examining the table we find that it gives functions only for angles measured in degrees. We shall now study its use in finding functions of angles measured in degrees and minutes.

Example 1. To find the sin 35° 13′.

Looking at the table you find two columns marked degrees, the first and last columns. Beginning at the top the first column contains all of the angle from 0 to 45 degrees. Beginning at the bottom the last contains all of the angle from 45 degrees to 90 degrees reading upward. From the work called for in Art. **235** explain why the table may be made in this way, the names at the top belonging to the angles of the first column, and the names at the bottom belonging to the angles of the last column.

Running down the first column you find 35°. Looking in the second column (marked sin ϕ at the top) in the same line we find

$$\sin 35° = .574.$$

Also $$\sin 36° = .588.$$

Our angle lies between these two angles, so we argue that its sine must lie between .574 and .588. We also notice that as the angle increases 1°, which is 60′, the sine of the angle increases .014. Therefore for an increase of 13′ in the angle we assume that there will be an increase of $\frac{13}{60}$ of .014 which is .003. Adding this to .574, we have .577. Therefore

$$\sin 35° 13' = .577.$$

Example 2. To find the cos 72° 44′.

Since this angle is more than 45°, run up the last column until you come to 72°. Look along the bottom line until you come to the column marked cosine. In this column in a line with 72°, you will find

$$\cos 72° = .309.$$

Also $$\cos 73° = .292.$$

Our angle lies between 72° and 73°. Our argument now is, since as the angle increases 60′, its cosine decreases .017, therefore as the angle increases 44′ the cosine will decrease $\frac{44}{60}$ of .017, which is .012. Subtracting this from .309, we have .297.

Therefore $\cos 72° 44' = .297.$

When the remainder in taking the fractional part for the increase or decrease is more than .0005, we use it as .001. When it is less than .0005 we do not use it. Do not carry out the decimal farther than the table that you are using.

Exercise. Find the following:

sin 54° 28′.	cos 63° 32′.
cos 22° 8′.	tan 56° 56′.
tan 47° 34′.	sin 15° 18′.

238. The next work is to see how from the tables we can find the number of degrees and minutes in an angle if we know its function.

Example 1. If $\sin \phi = .475$, find ϕ.

Running down the column marked $\sin \phi$, we find

$$\sin 28° = .469.$$
$$\sin 29° = .485.$$

Since .475 lies between these functions, the angle must be between these angles. So our argument is, since as the function increases .016, our angle increases 60′, therefore as the function increases .006, the angle will increase $\frac{.006}{.016}$ of 60′, which is $\frac{3}{8}$ of 60′, which is 22′. Therefore

$$\phi = 28° 22'.$$

Example 2. If $\cos \phi = .621$, find ϕ.

Looking in the column marked $\cos \phi$, we find

$$\cos 51° = .629.$$
$$\cos 52° = .616.$$

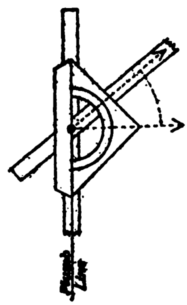
Since .621 lies between the two functions, our angle must lie between these two angles. So our argument is: When the function decreases .013, our angle increases 60′, therefore when our function decreases .008, our angle increases $\frac{.008}{.016}$ of $60'$ which is $\frac{1}{2}$ of $60'$, which is $30'$. Therefore $\phi = 51° 30'$.

Find ϕ in the following:

$\sin \phi = 0.885.$	$\cos \phi = 0.40.$
$\cos \phi = 0.969.$	$\tan \phi = 0.79.$
$\tan \phi = 8.00.$	$\sin \phi = 0.521.$

239. Now that we are able to find these six ratios from our table, we have six equations which we can use as formulæ, when we have taken proper measurements.

The following simple device is suggested for measuring angles, when a surveyors' transit cannot be had.

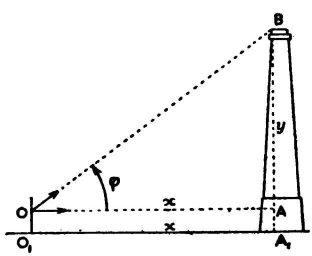


Fasten together two yard sticks or foot rules so that they will turn as on a hinge. To the rivet hang a string with a weight on it (a plumb line). With thumb-tacks fasten your triangular protractor to one of the sticks, the long side running along the stick, and its center directly on the center of the rivet.

To determine the height of an object, as A_1B in the figure below, hold the instrument at O and measure $\angle AOB$. A stick or pole should be used to support the instrument, and the distance O_1O should be measured, to be added to AB.

The distance O_1A_1 which is equal to OA can be measured. Thus we shall have ϕ and x, to find y in the formula,

$$\frac{y}{x} = \tan \phi, \qquad y = x \tan \phi.$$



Example 1. Suppose that angle ϕ is $23° 16'$, and distance OA is 300 feet. What is the height of AB?

Given $x = 300$, and $\phi = 23° 16'$. To find y.

Selecting the function which contains the parts we have given and also the part which we are trying to find, we choose

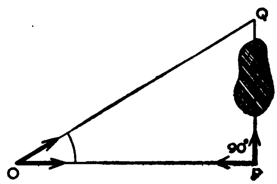
$$\tan \phi = \frac{y}{x}.$$

Solving for y, $y = x \tan \phi$.
Substituting values: $y = 300.\tan 23° 16'$.
$y = 300 \times 0.430$ (finding value of tan 23° 16′ from the table).
$= 129.0$ number of feet in height AB.

Check this by making a careful drawing on cross section paper and comparing results.

Example 2. Suppose that you wished to find the distance from one point to another between which there is an obstruction. The following measurements are suggested.

Let the points be P and Q. Adjust your sticks at right angles with the center of the protractor on point P, and one of the arms pointing in the direction of Q. From P in the direction of the other arm measure any convenient length PO. At point O adjust your sticks so that one will point in the direction of point P and the other in the direction of point Q. Read the measurement of the angle at O.



Suppose that the distance OP is 150 feet and the angle at O is 52° 29′. What is the distance PQ?

Let x = the measured distance OP.
Let y = the required distance PQ.
Given $x = 150$, and $\phi = 52° 29'$. To find y.

Selecting the function which contains the given parts and the parts that we wish to find:

$$\tan \phi = \frac{y}{x}.$$

Solving for y, $y = x \tan \phi$.

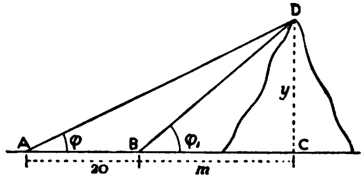
Substituting the given values, $y = 150 \times \tan 52° 29'$.
Finding value of $\tan \phi$, $y = 150 \times 1.302$.
$= 195.3$ number of feet in distance PQ.

Check by making careful drawing on cross-section paper and comparing results.

Example 3. Suppose that you cannot go to the object whose height you wish to know.

The following figure suggests measurements to take:

$AB = 20$ chains.
$\phi = 25°$.
$\phi_1 = 40°$.



Graphic Solution. On your paper lay off a distance 20 units. At each end draw the measured angles as shown in the figure. The arms will meet at point D. Prolong AB and drop a perpendicular from D. Measure CD.

Solution by Functions. Let the unknown distance BC be m chains and let CD be y chains. Then we have:

From $\triangle BCD$, $\frac{y}{m} = \tan 40°$.

From $\triangle ACD$, $\frac{y}{20 + m} = \tan 25°$.

Solve for y and m.

Ans. $m = \frac{20 \tan 25°}{\tan 40° - \tan 25°}$;

$y = \frac{20 \tan 25° \tan 40°}{\tan 40° - \tan 25°}$.

The above are given as suggestions. The student can work out many ways of finding distances which cannot be measured directly.

240. Exercises. In the following make drawings, solve for the parts called for by making use of the trigonometric functions and check by the drawings.

Here x, y, r, ϕ are used as in the figure of Art. **234**.

1. $x = 27.3$, $\phi = 59° 13'$, find y and r.
2. $r = 1.57$, $y = 1$, find ϕ and x.
3. $y = 27.33$, $r = 67.1$, find ϕ and x.
4. $y = 256$, $\phi = 35° 57'$, find x and r.

The following problems are given to show the uses of trigonometric functions. The best problems for the student to solve are those for which he has done his own measuring, as they have a personal interest.

5. At a horizontal distance of 120 feet from the foot of a steeple the angle of elevation of the top of the steeple is 60° 33′. Find the height of the steeple.

6. A train is running at the rate of 30 miles an hour. The mail pouch is thrown from it at right angles to the direction of the train. The pouch takes a direction which makes an angle of 30 degrees with the direction of the train. With what speed was the pouch thrown?

7. Two forces act at right angles to one another. One is 20 lbs. and the other is 35 lbs. What angle will the resultant force make with the greater of the two forces?

8. Two forces acting at right angles give a resultant force of 32.5 pounds. If one of the forces makes an angle of 27° 38′ with the resultant force, what is the number of pounds in each force?

9. Two forces, of which the first is 47 pounds less than the second, act at right angles to each other. Their resultant force is 65 pounds. Find the number of pounds in each force, and the angle which the resultant force makes with each.

10. Rain-drops are falling vertically with a speed of 100 miles an hour as they pass the windows of a train which is running 30 miles an hour. How will they seem to fall to a person in the train?

11. The tower of Pisa is 172 feet high and inclines 11 feet and 2 inches from the perpendicular. What is the angle of inclination?

241. Ratio and Proportion Involving Variables.

Note. The student may review First Course, pages 245 to 255, before going on with the work.

One of the simplest statements of the relation of two variables is

$$y = ax,$$

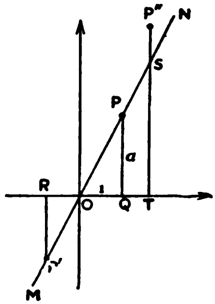
or, in ratio form,

$$\frac{y}{x} = a.$$

If we regard x and y as the coördinates of a variable point, this statement calls for all points such that the ratio of ordinate to abscissa is the fixed number a.

In mathematical language we are to find the locus of points whose coördinates satisfy the equation

$$\frac{y}{x} = a.$$



To prove that this locus is a straight line.

Locate point P by counting $x = 1$, and $y = a$. This is one point of the locus. Draw a straight line MN through this point and the origin, which is another point of the locus.

To prove that MN is the locus of points whose coördinates will always have the ratio a.

Analysis. In order to prove that this line is the locus required, we must prove that any point on the line has coördinates which satisfy the equation, and that any point not on the line has coördinates which will not satisfy the equation.

Let P' be any other point on the line other than point P. Let its coördinates be x', y'.

We must prove that $\frac{y'}{x'} = a.$

Proof. The triangles $P'OR$ and OQP are similar. Prove this.

Therefore $y' : x' = a : 1.$ Why?

Therefore the coördinates of P' satisfy the requirement that they have the ratio a.

Let P'' be any point not on the line MN. Let its coördinates be x'', y''.

We must prove that $\frac{y''}{x''}$ does not equal a.

Let y'' cut line MN at point S. The coördinates of point S are (x'', ST).

Since point S is on the line MN: $\frac{ST}{x''} = a$. Why?

Therefore $\frac{y''}{x''}$ does not equal a unless y'' equals ST. (Why?)

Therefore the coördinates of P'' do not satisfy the condition that their ratio is a.

So we have proved the theorem:

242. Theorem XXII. *The graph of the equation* $y = ax$ *is a straight line through the origin and point* $(1, a)$.

243. Theorem XXIII. *The graph of the equation* $y = ax + b$ *is a straight line.*

This equation is but the statement that for every value of x the value of y is b units more than the value of y in the equation $y = ax$.

Since the graph of $y = ax$ is a straight line, the graph of $y = ax + b$ is a straight line parallel to it, and b units higher or lower, depending on the sign of b.

Exercises. Draw the graph of the following:

1. $y = \frac{1}{2}x$. 2. $y = \frac{1}{2}x + 3$.
3. $y = 3x$. 4. $y = 3x - 5$.
5. $y = \frac{3}{4}x - 1$. 6. $2x - 2y = 5$ (rewrite as $y = x - 2\frac{1}{2}$).
7. $-3x + 7y = 14$. 8. $4x - 2y = -7$.

9. A tree, now b inches in diameter, grows at the rate of a inches a year. How thick will it be in x years? If the thickness is y, show that $y = ax + b$. Draw the graph when $a = 0.5$ and $b = 4$.

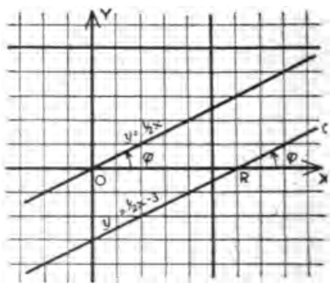
244. Since the inclination of the line $y = ax + b$ to the axis of the abscissas is the same as that of the line $y = ax$, we can readily find the angle which the graph of any linear equation makes with the axis of abscissas. Since for any straight line through the origin we have

$$\frac{y}{x} = a,$$

and since $$\tan \phi = \frac{y}{x},$$

therefore $$\tan \phi = a.$$

So that, knowing a, we can find value of ϕ from the tables.



Definition. The value of $\tan \phi$ is called the **slope** of the line.

Example. Take the line $y = \frac{1}{2}x - 3$.

This is parallel to the line $y = \frac{1}{2}x$.

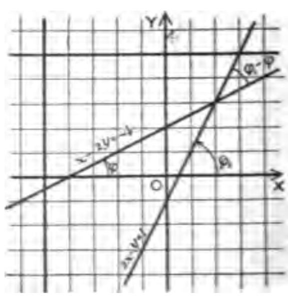
Therefore $\tan \phi = \frac{1}{2}$. Why?

$$\phi = 26° 34'.$$

Therefore the graph of $y = \frac{1}{2}x - 3$ makes an angle of $26° 34'$ with the axis of the abscissa.

Exercises. Determine the angle which the graphs of the following equations make with the axis of abscissas.

1. $2x - 3y = 7$. Rewrite as $y = \frac{2}{3}x - \frac{7}{3}$.
2. $-5x + y = 1$.
3. $4x - 16y = 48$.
4. $7x - y = -14$.



245. Angle between Two Lines. If two lines are drawn with reference to the same axes, we can easily find the angle which they make with one another. In the adjacent figure the lines are the graphs of the equations

$$x - 2y = -4,$$
$$2x - y = 1.$$

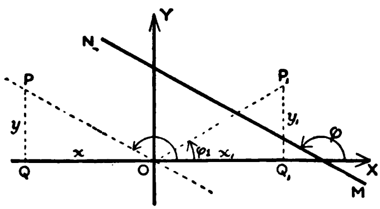
Let ϕ be the angle which graph of $x - 2y = -4$ makes with the axis of abscissas, and let ϕ_1 be the angle which the graph of $2x - y = 1$ makes with the axis of abscissas.

Then $\phi_1 - \phi$ is the angle between the two lines. (Art. **65.**)

Find the angle which the lines of the above exercises make with one another, by first finding the angles ϕ and ϕ_1. Check by measurement of an accurately drawn figure.

When $\tan \phi$ is positive, that is, when a is positive ϕ is acute and can easily be found from the tables.

When $\tan \phi$ is negative, we shall have to examine a little farther, since there are no negative numbers in our table.



We have given the line MN making the angle ϕ with the axis of the abscissa, ϕ being greater than a right angle. We can find an angle less than a right angle which has a tangent whose absolute value is the same as that of angle ϕ.

Analysis. In order to do this we draw through the origin a line parallel to MN. On it mark point P. Draw angle ϕ_1 equal to the supplement of ϕ. Make OP_1 equal to OP.

From this it is readily proved that

$$y = y_1, \quad \text{and} \quad x = -x_1.$$

Therefore $$\frac{y}{x} = -\frac{y_1}{x_1}. \qquad \text{Why?}$$

That is $$\tan \phi = -\tan \phi_1.$$

We have shown that the tangent of an angle is equal to the negative tangent of its supplement. To find an angle whose tangent is negative, take the supplement of the angle whose tangent is numerically the same, but positive.

246. Your attention is especially called to the use of the negative sign in the above. Students are too much inclined to say that an expression is a negative number because they see a negative sign written before it. In the expression $x = -x_1$, by the figure you see readily that x_1 is not negative. Neither does this algebraic expression say that it is negative. In fact it states nothing as to the real direction of either x or x_1. It merely states that x is of the opposite direction to x_1. The student should fix in mind this fact, that when a negative sign precedes a general number, it does not indicate that the number is negative, but that the expression is the negative of whatever particular value may be given to the general number.

$\tan \phi = -\tan \phi_1$ does not state that $\tan \phi_1$ is negative. We see that in the above figure, $\tan \phi_1$ is positive. The expression states that $\tan \phi$ is the negative of $\tan \phi_1$. Compare the other functions of angle ϕ with those of ϕ_1.

247. Negative Slope. Given the equation

$$4x + 3y = 15.$$

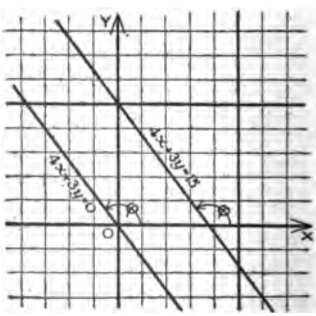
To make the graph and find its inclination to the axis of the abscissa.

Writing this equation in the form $y = ax + b$, we have

$$y = -\frac{4}{3}x + 5.$$

The equation of the line through the origin parallel to the graph of this line is

$$y = -\frac{4}{3}x \text{ or } \frac{y}{x} = -\frac{4}{3}.$$



Draw the graph of this by counting -3 in the direction of the axis of the abscissa, and 4 in the direction of the axis of the ordinate. Draw a line through this point and the origin.

Since for every point on the

graph of the equation $y = -\frac{4}{3}x + 5$, y is 5 units more than for the corresponding point on the line parallel to it through the origin, for the point corresponding to the origin we locate the point (0, 5), and for the point (− 3, 4), we locate (− 3, 9). Draw a line through these points and we have the graph of the given equation.

To find the value of angle ϕ.

From the equation we have

$$\frac{y}{x} = -\frac{4}{3},$$

that is

$$\tan\phi = \frac{y}{x} = -\frac{4}{3}$$
$$= -1.33+.$$

Looking in the table for the angle whose tangent is 1.33, we find tan 53° = 1.33. Since the tangent of the angle is the negative of the tangent of its supplement, tan 127° = − 1.33.

Therefore the line makes an angle of 127° with the axis of the abscissa.

248. Exercises. In the following make graphs, find the angle which each line makes with the axis of the abscissa, and the angle which they make with one another. Solve for x and y by algebra and note correspondence of values of x and y with coördinates of point of intersection of the lines.

1. $2x + 3y = 7$, $3x + 4y = 10$.
2. $2x - 3y = 10$, $5x + 2y = 6$.
3. $2x - 10y = 15$, $2x - 4y = 18$.
4. $3x + y = 4$, $x + 3y = -2$.
5. $3 - 15y = -x$, $3 - 15y = 4x$.
6. $6y - 10x = 14$, $y - x = 3$.
7. $2y + 5 = 7x$, $x + 3y = -42$.
8. $-5 - 3x = 5y$, $2x + y = 6$.

249. Summary and Questions.

Part I—Ratio and Proportion.

Definition. Ratio, proportion, commensurable and incommensurable numbers, antecedent, consequent, mean proportional.

Pencil of lines.

Experiment by measurement to bring out theorems.

Theorem I. If a pencil of lines is cut by a pencil of parallels, the corresponding segments are in proportion.

Remark. On the generalization of this theorem for incommensurable cases.

Theorem II. If four quantities are in proportion, they are in proportion by inversion.

Theorem III. If four quantities are in proportion, they are in proportion by alternation.

Theorem IV. In a series of equal ratios the sum of an antecedent and its consequent is to the consequent as the sum of any other antecedent and consequent is to the consequent.

Theorem V. In a series of equal ratios the difference between any antecedent and its consequent is to the consequent, as the difference between any other antecedent and consequent is to the consequent.

Theorem VI. If four quantities are in proportion the sum of an antecedent and its consequent is to their difference as the sum of the other antecedent and consequent is to their difference.

Theorem VII. In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

Problems I and II. To divide a line in any given ratio.

Definition of internal and external division.

Theorem VIII. If a pencil of lines is cut by a pencil of parallels the segments of the lines of the pencil (cut from the vertex) are proportional to the corresponding segments of the parallels.

Theorem IX. If a pencil of lines is cut by a pencil of parallels, the corresponding segments of the pencil of parallels are in proportion.

Theorem X. A line parallel to the side of a triangle divides the other two sides proportionally.

Theorem XI. A line which divides two sides of a triangle proportionally is parallel to the third side.

Theorem XII. If an angle of a triangle is bisected by a line cutting the opposite side, then the opposite side is divided internally into segments proportional to the other two sides.

Theorem XIII. If the exterior angle of a triangle is bisected by a line cutting the opposite side, then the opposite side is divided externally into segments proportional to the other two sides.

Part II—Similar Figures.

Definition. Similar systems of points and similar figures.

Corollary 1. If two triangles are similar their corresponding sides are proportional.

Corollary 2. If two triangles are similar they are mutually equiangular.

Theorem XIV. Two triangles are similar if they have two angles of the one equal respectively to the two angles of the other.

Corollary 1. Two triangles are similar if their corresponding sides are parallel each to each, or are perpendicular each to each.

Corollary 2. Equilateral triangles are similar.

Corollary 3. Two right triangles are similar if they have an acute angle of one equal to an acute angle of the other.

Theorem XV. Corresponding altitudes of similar triangles are proportional to any two corresponding sides.

Theorem XVI. If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, the triangles are similar.

Theorem XVII. If two triangles have their corresponding sides proportional they are similar.

Theorem XVIII. If two polygons are similar they can be divided into the same number of similar triangles, similarly placed.

Corollary 1. If two polygons are similar, they have their corresponding sides proportional and their corresponding angles equal.

Corollary 2. The perimeters of similar polygons have the same ratio as their corresponding sides.

Theorem XIX. If two polygons are mutually equiangular and have their corresponding sides proportional, they are similar.

Theorem XX. If a pencil of lines is cut by a circumference, the product of the segments (cut from the vertex) is constant.

Corollary. A tangent is a mean proportional between the segments cut on any other ray of the pencil.

Theorem XXI. If a perpendicular is dropped from the vertex of the right angle of a right-angled triangle, to the hypotenuse,

(a) it divides the triangle into two similar triangles each similar to the original triangle;

(b) either side of the triangle is a mean proportional between the hypotenuse and the adjacent segment into which the hypotenuse is divided;

(c) the perpendicular is a mean proportional between the segments into which the hypotenuse is divided.

Problem V. To construct a mean proportional between two given line segments.

Corollary. To construct a square equal to a given rectangle.

Part III—Trigonometric functions.

Opening discussion and illustration.

Definition of trigonometric functions.

Tables. Explanation of use, and exercises in use.

Ratio and Proportion involving variables.

Theorem XXII. The graph of the equation $y = ax$ is a straight line through the origin and point $(1, a)$.

Theorem XXIII. The graph of a linear equation is a straight line.

1. Into what three parts is this chapter divided?

2. What is the ratio of one number to another.

3. Illustrate commensurable and incommensurable ratios of line-segments.

4. State some theorems which may arise when you have four quantities in proportion.

5. Explain how they are proved by proving that if a pencil of lines is cut by a pencil of parallels, the corresponding segments are in proportion.

6. Describe what is meant by similar systems of points.

7. What are similar triangles?

8. State the theorems on congruent triangles and similar triangles in corresponding pairs.

9. Are there any propositions in this chapter which were proved without using any of the preceding theorems? If so, state them.

10. Give the six trigonometric functions and tell why they are called the functions of an angle.

11. To what practical use can the trigonometric functions be put?

12. Do you regard the work of this chapter any more practical than that of the preceding chapter?

13. Look through the list of exercises and select those that you regard as practical in nature.

14. What is the graph of a linear equation?

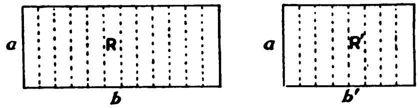
Most of the theorems of this chapter are selected from Book IV of Euclid's Geometry.

CHAPTER VII

PART I—AREAS OF RECTILINEAR FIGURES. MENSURATION OF CIRCLES. PART II—DIVISION OF A PERIGON. PART III—INCOMMENSURABLE CASES

PART I—AREAS OF RECTILINEAR FIGURES

250. Theorem I. *Two rectangles having equal altitudes are to each other as their bases.*



Given the rectangles R and R' with equal altitudes a, and with unequal bases b and b', respectively.

To prove that $\dfrac{\text{area of } R}{\text{area of } R'} = \dfrac{b}{b'}$.

Analysis. We can prove this theorem by expressing the areas of R and R' in terms of an area-measuring unit, and the lengths of b and b' in terms of a linear measuring unit, and showing their ratios to be equal. To do this divide b and b' by the same unit of measure u. Let it go m times into b, and n times into b'. At the points of division draw lines perpendicular to b; these will divide R into m equal rectangles, and R' into n equal rectangles, whose area is r. Now by finding the lengths of b and b' in terms of u, and from this finding their ratio, then by finding the areas of R and R' in terms of r, and finding their ratio, we can prove the proportion in the theorem.

Proof. The rectangles into which R and R' are divided are equal. Why?

$$b = mu,$$

and $$b' = nu.$$ Why?

$$Area\ of\ R = mr,$$

$$Area\ of\ R' = nr.$$ Why?

So that $$\frac{b}{b'} = \frac{mu}{nu} = \frac{m}{n}.$$ Why?

Also, $$\frac{Area\ of\ R}{Area\ of\ R'} = \frac{mr}{nr} = \frac{m}{n}.$$ Why?

$\therefore$ $$\frac{Area\ of\ R}{Area\ of\ R'} = \frac{b}{b'}.$$ Why?

State the theorem proved.

Discussion. In the figure, and in the proof, it was assumed that b and b' have a common unit of measure.

This proof would not hold if the base of one rectangle were equal to the side of a square and the base of the other rectangle were equal to the diagonal of the same square. Why not?

Neither could we use this proof if the base of one rectangle were equal to the circumference of a circle and the base of the other were equal to the radius of the same circle. Why not?

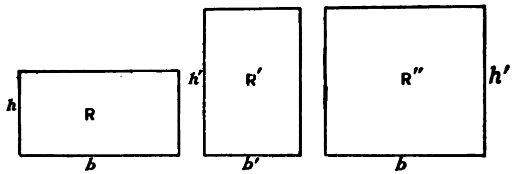
The proof that this theorem is true for such cases, commonly known as the incommensurable cases, is given after the treatment of theory of limits at the end of this chapter. When this theorem is proved for the incommensurable cases, all the proofs of following theorems which depend upon this theorem will hold for incommensurable cases without further discussion.

Corollary 1. *Rectangles having equal bases are to each other as their altitudes.*

Corollary 2. *Parallelograms having equal altitudes are to each other as their bases, and having equal bases are to each other as their altitudes.* Art. **94**, Cor. 2.

Corollary 3. *Triangles having equal altitudes are to each other as their bases, and having equal bases are to each other as their altitudes.*

251. Theorem II. *Two rectangles are to each other as the product of their bases and altitudes.*



Given two rectangles R **and** R'**, with bases** b **and** b' **and altitudes** h **and** h'**.**

To prove that $\dfrac{\text{area of } R}{\text{area of } R'} = \dfrac{bh}{b'h'}$.

Analysis. Since the only propositions that we have proved deal either with rectangles of equal bases or of equal altitudes, we shall form such rectangles. We can get these by drawing a new rectangle R'', whose base is b and whose altitude is h'. Now by writing the ratios of each of the given rectangles to this new rectangle, we can, by dividing, eliminate the new rectangle and have the ratio called for in the proposition.

Proof. $$\frac{R}{R''} = \frac{h}{h'}. \qquad \text{Why?}$$

$$\frac{R'}{R''} = \frac{b'}{b}. \qquad \text{Why?}$$

Dividing the first equation by the second,

$$\frac{R}{R'} = \frac{bh}{b'h'}. \qquad \text{Why?}$$

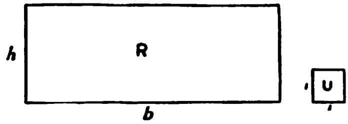
State the theorem proved.

Exercise. Find the ratio of the rectangles whose dimensions are:

(*a*) 2 by 3, and 5 by 6. (*b*) 5 by $\sqrt{2}$, and $2\sqrt{2}$ by 20.

252. Theorem III. *The area of a rectangle is equal to the product of its base by its altitude.*

Note. By this we mean that the number of square units in the area is equal to the number of linear units in the base multiplied by the number of linear units in the altitude.



Given the rectangle R, with base b units long and altitude h units.

To prove that the area of R equals bh.

Analysis. To find the area of R we must find its ratio to a unit of area; we select the square unit U whose dimension is 1. We have now but to find the ratio of rectangle R to rectangle U.

Proof. $\frac{R}{U} = \frac{bh}{1 \cdot 1}$. According to Theorem II.

The ratio of R to U is the area of R;

$\therefore$ *area of* $R = bh$.

State the theorem proved.

Corollary 1. *The area of a parallelogram is equal to the product of its base by its altitude.*

Corollary 2. *The area of a triangle is equal to one-half the product of its base by its altitude.*

Corollary 3. *The area of a trapezoid is equal to one-half the product of the sum of its bases by its altitude.*

253. Exercises. (Draw careful figures on cross-ruled paper.)

1. Find the area of the triangle whose vertices are the points $(-3, 2)$, $(1, 2)$, $(\frac{1}{2}, 5)$.

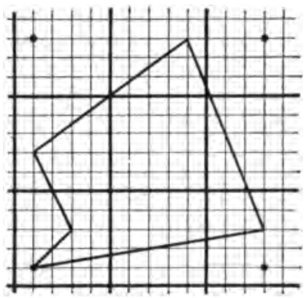
2. What kind of a figure is formed by the points $(-1, -3)$, $(7, -3)$, $(9, 4)$, $(1, 4)$. Compute the area.

3. What kind of a figure has for its vertices the points (– 10, – 7), (12, – 7), (5, 2), (– 2, 2)? Compute the area.

4. Compute the area of the figure the coördinates of whose vertices are (– 10, – 7), (12, – 7), (5, 2), (1, 3), (– 2, 2).

5. Compute the area of the figure whose vertices are the points (– 10, – 7), (14, 9), (12, – 7), (5, 2), (1, 3), (– 2, 2).

6. How many acres in this field if the side of one small square is 10 rods?



Do this by starting with the rectangle whose corners are marked by the dots and cutting off triangles until you have left the area required.

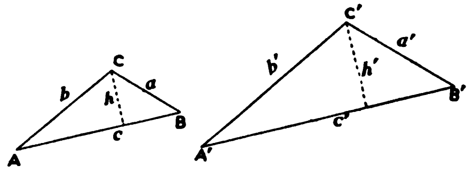
Study out other methods for getting this area.

7. What is the altitude of an equilateral triangle whose perimeter is 15 units and area 60 square units? Solve when the perimeter is p units and the area a square units.

8. In an isosceles trapezoid the altitude is 12 units and one of the equal sides is 13 units. What is its area? Solve when the altitude is a units and one of the equal sides is s units.

9. A rectangle which is twice as wide as it is high is inscribed in a circle of radius 10 units. Calculate the dimensions and the area of the rectangle. Solve when the base is n times the height, and the radius is r.

254. Theorem IV. *Two similar triangles are to each other as the squares of two corresponding sides, or as the squares of two corresponding altitudes.*



Given triangles ABC and $A'B'C'$, with the corresponding sides a and a', b and b', c and c', and altitudes h and h'.

To prove that $\frac{\text{area of } \triangle ABC}{\text{area of } \triangle A'B'C'} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \frac{c^2}{c'^2} = \frac{h^2}{h'^2}$.

Analysis. Since the propositions which we have had on areas of triangles dealt with the bases and altitudes, we use these. Now by writing the area of each triangle and finding the ratio between them, and remembering that $\frac{h}{h'} = \frac{b}{b'} = \frac{a}{a'} = \frac{c}{c'}$, we shall be able to prove the equality of the ratios called for in the theorem. Let T denote the area of $\triangle ABC$, and T' the area of $\triangle A'B'C'$.

Proof. $T = \frac{1}{2} bh$. Why?

$T' = \frac{1}{2} b'h'$. Why?

$\therefore$ $\frac{T}{T'} = \frac{\frac{1}{2} bh}{\frac{1}{2} b'h'} = \frac{bh}{b'h'}$. Why?

But since the triangles are similar,

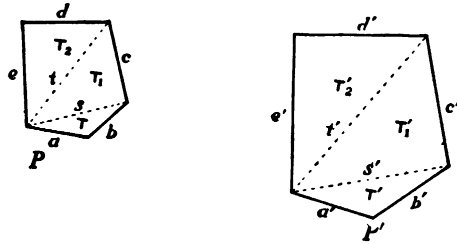
$$\frac{h}{h'} = \frac{b}{b'} = \frac{a}{a'} = \frac{c}{c'}. \quad \text{(Art. 221.)}$$

Substituting in the above equation, we have

$$\frac{T}{T'} = \frac{b}{b'} \cdot \frac{h}{h'} = \frac{b^2}{b'^2} = \frac{h^2}{h'^2} = \frac{a^2}{a'^2} = \frac{c^2}{c'^2}. \quad \text{Why?}$$

State the theorem proved.

255. Theorem V. *Two similar polygons are to each other as the squares of any two corresponding sides.*



Given the polygons P and P' with the corresponding sides a and a', b and b', c and c', and so on.

To prove that $\frac{P}{P'} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \frac{c^2}{c'^2}$, *and so on.*

Analysis. Divide the polygons into similar triangles as shown in the figure. Show that the ratios of corresponding triangles are equal, by showing each equal to $\frac{a^2}{a'^2}$. Place each ratio equal to this, clear fractions and add together the equations. From this find the ratio of the sums of triangles and hence the ratio of the polygons.

Proof. $\frac{T}{T'} = \frac{a^2}{a'^2}$. Why? $\therefore\ Ta'^2 = T'a^2$.

$\frac{T_1}{T'_1} = \frac{s^2}{s'^2} = \frac{a^2}{a'^2}$. Why? $\therefore\ T_1a'^2 = T_1'a^2$.

$\frac{T_2}{T'_2} = \frac{t^2}{t'^2} = \frac{s^2}{s'^2} = \frac{a^2}{a'^2}$. Why? $\therefore\ T_2a'^2 = T_2'a^2$.

Adding: $(T + T_1 + T_2)\, a'^2 = (T' + T_1' + T'_2)\, a^2$. Why?

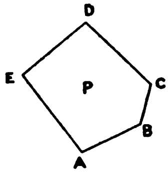
$$\therefore \quad \frac{T + T_1 + T_2}{T' + T'_1 + T'_2} = \frac{a^2}{a'^2}, \qquad \text{Why?}$$

or,

$$\frac{P}{P'} = \frac{a^2}{a'^2}.$$

State the theorem proved.

256. Problem I. *To construct a polygon similar to a given polygon and having a given ratio to it.*



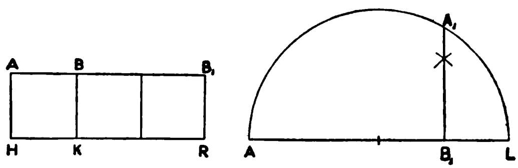
Given the polygon P.

To construct a similar polygon n times as large.

We shall give the work for $n = 3$.

Analysis. Since these polygons must have the same ratio as the squares of any two corresponding sides, we first find the side of a square which is three times the square on side AB, say. On this we construct a polygon similar to P. This will be the polygon required.

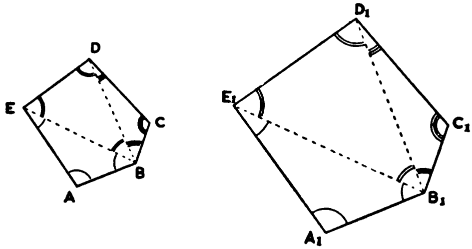
Construction.



(*a*) Construct a square, $AHKB$, whose side equals side AB of the given polygon. Then produce HK, making $HR = 3\,HK$. Rectangle HRB_1A will then equal $3\,\overline{AB}^2$.

(*b*) Construct A_1B_1 a mean proportional between HR and HA, as shown in the second figure (see Problem V, page 197), and we have $\overline{A_1B_1}^2 = 3\,\overline{AB}^2$. Therefore A_1B_1 is the side of the required polygon corresponding to side AB of the given polygon.

On A_1B_1 construct a polygon similar to the polygon P. Construct $\triangle\ A_1B_1E_1$ similar to $\triangle\ ABE$; then $\triangle\ B_1E_1D_1$ similar to $\triangle\ BED$; then $\triangle\ B_1D_1C_1$ similar to $\triangle\ BDC$.



Proof. $$\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{A_1B_1}^2} = \frac{\overline{AB}^2}{3\overline{AB}^2} = \frac{1}{3}.$$ Why?

257. Exercises.

1. A triangular field has sides of lengths 10 rods, 20 rods, and 25 rods. Construct a field half as large and of the same shape, on a scale of 5 rods to the inch.

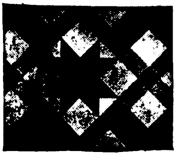
2. Compare the areas of two triangles formed by the bisector of an angle included between the sides of a triangle, whose lengths are 5 cm. and 10 cm. respectively.

3. Prove that the diagonals of a quadrilateral divide the quadrilateral into four triangles which form a proportion.

4. In copying the map of a triangular piece of ground from one paper to another it was decided to make the copy one-half as large as the map. How long must the sides of the new triangle be as compared with the first?

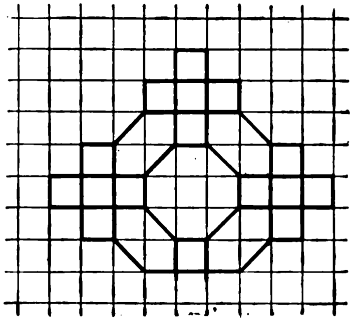
5. If in copying the map spoken of in the above exercise it was decided to make the new map one nth part as large as the first, how long would the sides be as compared with the first?

6. Construct the linoleum design illustrated in the adjoining figure so that the square containing the star shall have 16 times the area in the illustration.

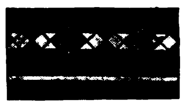


Calculate the area of the star in your figure.

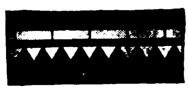
7. Construct parts of this field so that the area of the figures shall be five times as large as the illustration. Line



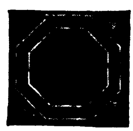
out the figure on cross-section paper ruled in squares of the proper size. Notice that, to compare the two figures, one of them should be turned through one eighth of a turn. What is the area of one of the octagons, and of one of the hexagons, if the side of one of the squares is a inches.



8. Construct part of the tile border making the areas of the figures six times as large as shown in the illustration.



9. Construct part of the tile border making the area of your drawing four times as large as that illustrated.



10. Construct a design similar to that of the adjacent figure having the area five times as large as the area illustrated.

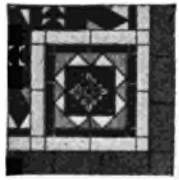
11. Construct parquetry border making the areas of the figures sixteen times the areas in the illustration.

Calculate the areas of the figures found in this design.

12. Construct this parquetry border design eight times as large as shown in the illustration.

Then measure the lines necessary and calculate the areas of the principal figures.

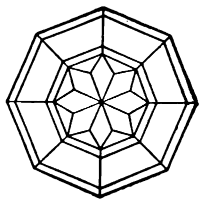
13. Construct this design for tile corner so that the square shall be sixteen times the size shown.



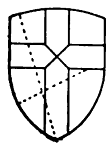
Calculate the area.

14. Construct part of this parquetry border so that the dark square shall be one inch on the side. Calculate the areas of all the figures lying within the large square.

15. Construct figures similar to the given figures but five times as large.

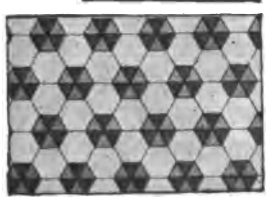


16. Construct the drawing of shield 4 inches high, 3 inches wide, and with bars $\frac{3}{4}$ of an inch wide.

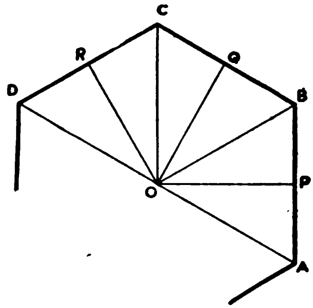


17. Construct this border four times as wide as shown in the illustration.

18. Construct part of this field spacing the parallel lines so that the hexagons shall be just half an inch on a side.



258. Theorem VI. *In a regular polygon the bisectors of the angles meet in a point equidistant from the sides, and the perpendicular bisectors of the sides meet in a point equidistant from the vertices of the polygon. These two points coincide.*



Given the regular polygon $ABCD \cdots$.

(*a*) *To prove that the bisectors of the angles meet in a point equidistant from the sides.*

(*b*) *To prove that the perpendicular bisectors of the sides meet in a point equidistant from the vertices.*

Analysis. (a) Draw the bisectors of $\angle$s A and B and suppose them to meet at O. Show that the bisector of $\angle C$ must go through O. (b) Show that the perpendicular bisectors of the sides all go through O. Let the student give the proof.

Definitions.

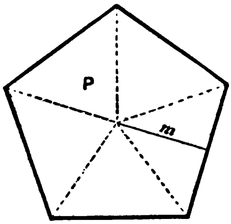
The lines OA, OB, OC and so on (figure above) are the **radii** of the regular polygon.

The lines OP, OQ, OR and so on are the **apothems** of the regular polygon.

Corollary. *The radii of a regular polygon are equal. The apothems of a regular polygon are equal.*

Exercise. A regular polygon of sixteen sides has each side a inches long and apothem b inches long. What is its area? Can you give any special values to a and b?

259. Theorem VII. *The area of a regular polygon is equal to the product of its perimeter by one-half of its apothem.*



Given the regular polygon P**, of area** a**, perimeter** p**, apothem** m**.**

To prove that $a = \frac{1}{2} pm$.

Analysis. To show this we show that the polygon can be divided into triangles the sum of whose areas equals $\frac{1}{2} pm$. Letting O be the center of the polygon, draw the radii r to the vertices, and prove that the sum of the areas of the triangles formed equals $\frac{1}{2} pm$.

Proof. Let the student give this proof.

260. Theorem VIII. *Two regular polygons of the same number of sides are similar.*

Suggestion. The student can prove this by showing a close connection with Art. **226.**

Corollary 1. *The perimeters of two regular polygons of the same number of sides are proportional to their sides, to their apothems, to their radii.*

Corollary 2. *The areas of two regular polygons of the same number of sides are proportional to the squares of their sides, to the squares of their apothems, to the squares of their radii.*

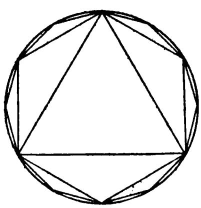
Exercise. The sides of two regular polygons are in the ratio 3 : 2, and the area of the first is 54 square inches. What is the area of the second?

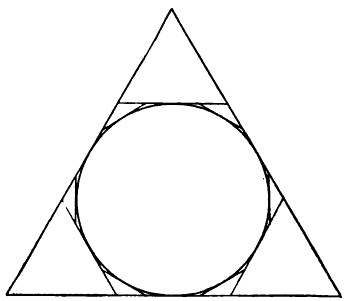
261. Areas of Circles.

To prove theorems on the circle we shall connect them with theorems on regular polygons, since those are the theorems which we have proved and to which they are most closely related.

This is a subject which gave the ancient Greek mathematicians much trouble. Of particular interest is the amount of time and energy spent on the problem of finding a square or other rectilinear figure whose area exactly equals the area of a given circle. This was called the problem of "squaring the circle." Closely related to this was the problem of finding the ratio of a circumference to its diameter, which attracted an equal amount of attention.

262. Examine and discuss the following figures.





The first figure is a circle with regular polygons inscribed in it, the second is a circle with regular polygons circumscribed about it. In either figure, in going from one polygon to the next the number of sides is doubled.

In the first figure the number of sides is constantly increasing. Is there any limit to the number of sides that the polygons may have?

Notice the length of each side. What is happening to it as you increase the number of sides?

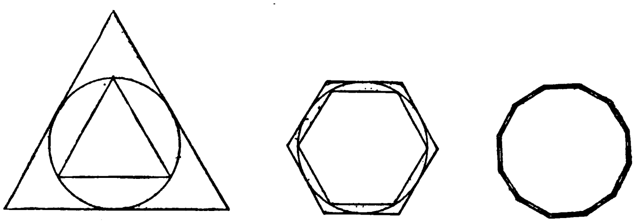
Notice the length of the perimeter. What is happening to it as you double the number of sides of the polygon? Could the perimeter ever coincide with the circumference of the circle by the process of doubling the number of sides?

Notice the area of the polygon. What is happening to the area as you increase the number of sides? Can the area of the polygon ever become the area of the circle?

Answer similar questions with reference to the second figure.

So we have these facts: In the inscribed polygon or the circumscribed polygon, as we increase the number of sides, the length of each side continually decreases. In the case of the inscribed polygon the perimeter and area are continually increasing. Why? In the case of the circumscribed polygon the perimeter and area are continually decreasing. Why?

263. Let us examine these changes when the polygons are inscribed in and circumscribed about the same circle.



In the first figure we have a circle, with an inscribed and a circumscribed equilateral triangle.

In the second figure we have a circle with inscribed and circumscribed regular hexagons. We have twice as many sides as in the first figure.

In the third figure we have a circle with inscribed and circumscribed regular dodecagons. We have again doubled the number of sides.

Suppose that we continue this process of doubling the number of sides. The areas of the inscribed polygons are steadily increasing and the areas of the circumscribed polygons are steadily decreasing. Consequently these areas are coming closer together, or, mathematically speaking, they are approaching a **common limit.** This common limit is called the **area of the circle.**

That the areas of the inscribed and circumscribed polygons come closer together as we increase the number of sides is also shown by the following table, which gives the areas. (Here the radius of the circle is r units.)

	Areas of Polygons	
Number of Sides.	Inscribed.	Circumscribed.
3	$1.29904\ r^2$	$5.19615\ r^2$
6	$2.59808\ r^2$	$3.46410\ r^2$
12	$3.00000\ r^2$	$3.21539\ r^2$
24	$3.10583\ r^2$	$3.15966\ r^2$
48	$3.13263\ r^2$	$3.14608\ r^2$
96	$3.13934\ r^2$	$3.14272\ r^2$
192	$3.14103\ r^2$	$3.14187\ r^2$

Exercise. Regular polygons of 12 sides are inscribed and circumscribed about a circle of 3 inches radius. Find by the table the difference of their areas. Do the same for polygons of 192 sides.

264. Now study the perimeters of the polygons. Just as in the case of areas, the perimeters of the inscribed polygons steadily increase and the perimeters of the circumscribed polygons steadily decrease, so that they come more and more nearly together, or **approach a common limit.** This common limit is called the **length of the circumference of the circle.**

Examine the following table and note that the perimeters of the inscribed and circumscribed polygons are becoming more nearly equal. (Diameter of circle is d units in this computation.)

	Perimeters of Polygons.	
Number of Sides.	Inscribed.	Circumscribed.
3	2.59808 d	5.19615 d
6	3.00000 d	3.46410 d
12	3.10583 d	3.21539 d
24	3.13263 d	3.15966 d
48	3.13934 d	3.14608 d
96	3.14103 d	3.14272 d
192	3.14145 d	3.14187 d

Exercise. Repeat the preceding exercise, calculating perimeters instead of areas.

265. The Number π. The number which these multipliers are approaching as we go on increasing the number of sides is usually expressed by the character π.

The approximate value of π is 3.1416, or less accurately, $3\frac{1}{7}$.

This symbol was first used in 1706. Its value has been found to more than 700 decimal places. The value of π to 15 places is

$$\pi = 3.14159\ 26535\ 89793\ \ldots.$$

Metius' value is $\pi = \frac{355}{113}$. (Error 1 : 1300000). Divide out the fraction and see how many places in the value of π come out correctly. Do the same with the value $\frac{22}{7}$.

266. In the preceding discussion we have made the following assumptions regarding the area and the perimeter of a circle.

Assumption I. *The area of a circle is the limit approached by the area of a regular circumscribed polygon, or by the area of a regular inscribed polygon, when the process of doubling the number of sides is steadily continued.*

Assumption II. *The length of the circumference of a circle is the limit approached by the perimeter of the regular circumscribed polygon, or the perimeter of the regular inscribed polygon,* *when the process of doubling the number of sides is steadily continued.*

Corollary 1. *The area of a circle may be calculated as closely as we please by calculating the area of a regular inscribed or circumscribed polygon having a sufficiently large number of sides.*

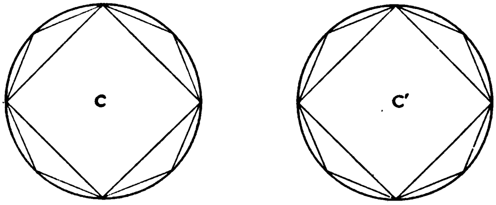
Corollary 2. *The circumference of a circle may be calculated as closely as we please by calculating the perimeter of a regular inscribed or circumscribed polygon having a sufficiently large number of sides.*

267. Definition. A variable magnitude is said to approach a fixed magnitude as a **limit** when the difference between the fixed and variable magnitudes becomes and remains arbitrarily small in absolute value.

In what follows we shall repeatedly use the following assumption:

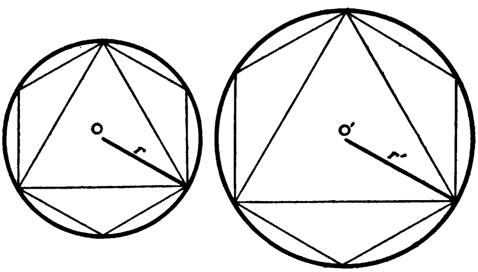
Assumption III. *If two variables are constantly equal and each approaches a limit, the limits are equal.*

Illustration. Suppose that we have two circles C and C' whose comparative sizes we do not know.



Suppose that in circle C we inscribe a polygon P. Then suppose that in C' we inscribe a polygon P' which we know to be of the same area as polygon P. Then suppose that we double the number of sides of each polygon, and find that again the new polygons are equal. Suppose that we continue this experiment again and again and always find polygon P' equal to polygon P. Could we do this if circle C' were less than circle C? You see readily that we could not, for if the circle C' were less than circle C it would stop the polygon P' from growing before the circle C stopped the polygon P from growing. So that since we have assumed that P' is always to remain equal to P, then C' could not be less than circle C. By a like discussion show that C' could not be greater than C. Therefore we assume that if the two variable polygons are to remain equal, their limits must be equal.

268. Theorem IX. *The ratio of the circumference to the diameter of a circle is constant.*



Given the circles whose centers are O and O', with diameters $2r$ and $2r'$, and circumferences c and c'.

To prove that $\frac{c}{2r} = \frac{c'}{2r'}$.

Analysis. Since we have studied the ratio of the perimeters of similar regular polygons, as compared to the ratio of the radii, we shall inscribe in the circles two similar regular polygons whose perimeters are p and p' respectively. We then have the proportion $\frac{p}{r} = \frac{p'}{r'}$. Double the number of sides of each of the polygons. State the proportion that exists between perimeters and radii of the new polygons. Continue this doubling, and stating ratios. Are these ratios always equal to each other? Will $\frac{p}{r}$ continue to approach $\frac{c}{r}$ and $\frac{p'}{r'}$ continue to approach $\frac{c'}{r'}$ as we double the number of sides? So can we state that $\frac{c}{r} = \frac{c'}{r'}$?

Proof. Let p and p' be the perimeters of inscribed equilateral triangles.

Then $\frac{p}{r} = \frac{p'}{r'}$. Why? See Art. **260,** Corollary 1.

Double the number of sides and let p and p' be the lengths of the new perimeters. Then

$$\frac{p}{r} = \frac{p'}{r'}. \qquad \text{Why?}$$

Continuing to double the number of sides,

$\frac{p}{r}$ is a variable which approaches $\frac{c}{r}$ as a limit.

$\frac{p'}{r'}$ is a variable which approaches $\frac{c'}{r'}$ as a limit.

$$\therefore \quad \frac{c}{r} = \frac{c'}{r'}. \qquad \text{Why?}$$

$$\therefore \quad \frac{c}{2r} = \frac{c'}{2r'}. \qquad \text{Why?}$$

State the theorem proved.

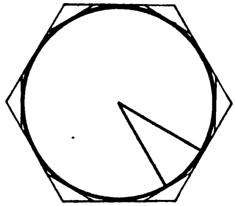
Corollary 1. *Letting π express the ratio $\frac{c}{d}$, we have* $\mathbf{c} = \pi \mathbf{d}$.

Corollary 2. *Since $c = \pi d$, and $d = 2r$, therefore* $\mathbf{c} = \mathbf{2}\pi \mathbf{r}$.

Corollary 3. *Two circumferences are proportional to their radii.*

Exercise. Calculate the perimeters of pipes whose diameters, in inches, are as follows: 6, 6.5, 7, 7.5, 8. Give results in tabular form, taking the approximate value of π, first $\frac{22}{7}$, then 3.1416.

269. Theorem X. *The area of a circle is equal to the product of its circumference by one-half of its radius.*



Given the circle C with circumference c and radius r.

To prove that area of C equals $\frac{1}{2}cr$.

Analysis. To prove this circumscribe a regular polygon P about the circle C. Let a_p denote the area of the polygon, and a_c the area of the circle. The perimeter of this polygon is p and its apothem is r, so that its area is $\frac{1}{2}pr$.

Therefore $a_p = \frac{1}{2}pr$.

Now as we continually increase the number of sides of the polygon, p will approach c, and a_p will approach a_c, as limits. This is the basis of the proof.

Proof. Since $p \doteq c$ as a limit,

then $\frac{1}{2}pr \doteq \frac{1}{2}cr$ as a limit.

Also $a_p \doteq a_c$ as a limit.

But $a_p = \frac{1}{2}pr$;

$\therefore$ $a_c = \frac{1}{2}cr$. Why? Art. **267.**

Corollary 1. *Since $a = \frac{1}{2}cr$, and $c = 2\pi r$, the formula for the area of the circle is*

$$\mathbf{a} = \pi \mathbf{r}^2.$$

Corollary 2. *The areas of two circles are to each other as the squares of their radii.*

270. Exercises.

1. Write the value of r in terms of a, also in terms of c.

2. If a circle has a radius of 10 units, find its circumference and its area. If its radius is 1.5; .001; 7.12.

3. If the area of a circle is 25π, what is its radius? What is its circumference?

Answer the same questions if area is 6.561. If .0289.

Draw a circle whose area shall be 9π square inches; 10π square inches.

4. Find the area of the ring between the circumference of two concentric circles whose radii are r_1 and r_2.

5. Using the answer to Exercise 4 as a formula, find the area between two circumferences whose radii are 4 and 10; whose radii are .7 and .06.

6. What is the area of a circle which is tangent to the circles mentioned in Exercise 4?

What is this area if radii of the circles are 4 and 10?

7. What is the area of a circle which is inscribed in a square of side s? What is the ratio of the circle to the square?

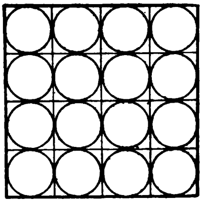
8. What is the area of a circle which is circumscribed about a square of side s?

What is the ratio of the circumscribed and inscribed circles?

9. What is the area of a circle which can be inscribed in an equilateral triangle of side s?

What is the area of a circle which can be circumscribed about this triangle?

What is the ratio of circumscribed and inscribed circles?

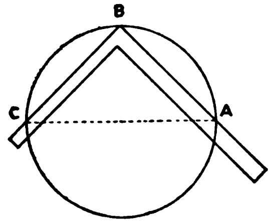


10. Compare the sum of the areas of the small circles in the adjacent figure with the area of the whole square and with the area of its inscribed circle.

11. If the diameter of the earth is taken as 42,000,000 feet, what is its circumference? Use Metius' value for π (see Art. **265**).

Taking the radius of the earth as 3960 miles, calculate the length of one degree of arc on a meridian. One sixtieth of this length is the nautical mile, or knot.

12. Explain the following process for finding the diameter of a pipe which shall have just half the carrying capacity of a given pipe.



The circle represents the opening of the given pipe. ABC is a carpenter's square so placed that $AB = BC$. Then AB is the diameter of the pipe required.

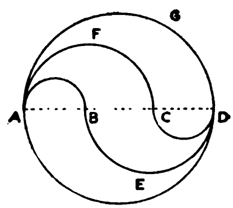
13. The diameters of two water pipes are 3 inches and 4 inches respectively. Find the diameter of a pipe which can carry as much water as both of them. Do this by algebra and by construction.

14. How nearly correct is the following proportion?

Let s be the side of a given square, and d the diameter of an equal circle.

Then approximately, $s : d = 8 : 9$.

Calculate d if $s = 10$, using the above proportion. What is the error?



15. Construct the adjoining figure:

Suggestion:

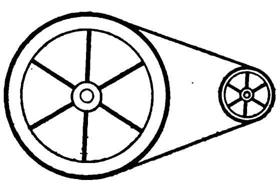
Take $AB = BC = CD = 1$ inch.

Calculate the area of the figure $ABEDCFA$ and of figure $AFCDGA$.

Calculate the areas using general numbers. Take

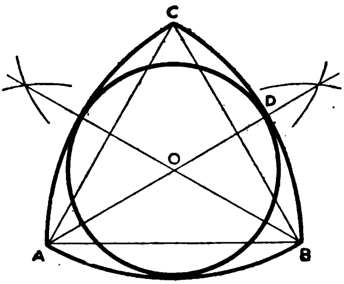
$$AC = 2a \text{ and } AB = CD = 2b.$$

16. Two wheels are connected by a belt, the diameters of the wheels being respectively 4 feet and 1 foot. If the large wheel turns 120 times a minute how fast does the small wheel turn?



17. If the wheels in Exercise 16 have radii equal to a feet and b feet respectively, and if the large wheel turns n times a minute, how fast does the small wheel turn?

18. Figure ABC is formed by 60° arcs, struck with radius AB. Construct the inscribed circle taking AB equal to 4 inches.



Show that O is the center of the equilateral triangle.

Compute the radius of the circle if $AB = 2a$.

Suggestion: Show that line $OA = \frac{2a}{3}\sqrt{3}$, and that line $OD = 2a - \frac{2a}{3}\sqrt{3}$

$= \frac{2a}{3}(3 - \sqrt{3})$. Compute the area of the circle.

Letting $a = 2$ *cm*, compute the radius and area of the circle.

Compute the lengths of the arcs AB, BC, CD.

19. Show that the ring formed by two concentric circles has an area equal to the area of a circle described on the line which is the chord of the larger circle and tangent to the smaller.

20. The front sprocket wheel has three times as many teeth as the rear sprocket wheel. The tires of the bicycle are 28 inches in diameter. How far will the bicycle go when the pedals make one complete revolution?

21. The wooden rims of the wheels in Exercise 19 are 26 inches in diameter and the hubs 2 inches in diameter. The spokes are fastened tangent to the hub. Calculate the length of a spoke.

Suggestion. You have here two concentric circles, and a line drawn from a point in the outer circle tangent to the inner circle. It is required to find the length of this line.

PART II—DIVISION OF THE PERIGON

271. In the theorems just discussed we have used inscribed and circumscribed polygons. We shall now investigate the construction of such figures with rule and compass.

This necessitates the division of the perigon into equal parts. The primary division of the perigon into equal parts by the use of rule and compass is possible only in certain special cases. Of course after it is once divided, it can be divided into a greater number of parts by a repeated bisection of the angles.

272. Problem II. *To bisect a perigon.*

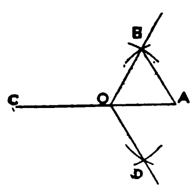
Bisect a perigon on the same principle that you would bisect any other angle.

By bisecting the straight angles thus formed you can divide the perigon into four equal parts.

By repeatedly bisecting angles you can divide the perigon into 2^n equal parts.

273. Problem III. *To trisect a perigon.*

This you have also learned to do, for you have learned to construct an equilateral triangle. The angle of this triangle is one-sixth of a perigon. The remaining angle of the straight angle is one-third of the perigon. Thus:



Angle BOC is $\frac{1}{3}$ of a perigon.
Angle COD is $\frac{1}{3}$ of a perigon.
Angle DOB is $\frac{1}{3}$ of a perigon.

The perigon then can be divided into $3 \cdot 2^n$ equal parts by the bisecting of these angles.

274. Problem IV. *To divide a perigon into five equal parts.*

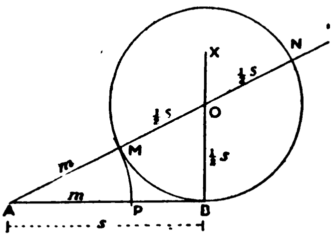
This construction is quite complicated, and we shall have to investigate the construction of the division of a line-segment into what is called *extreme and mean ratio.*

275. Definition. A line-segment is divided in extreme and mean ratio when one of the parts is a mean proportional between the other part and the whole line-segment.

In the figure below, if P is to divide segment AB in extreme and mean ratio we must have

$$AB : AP = AP : PB, \text{ or } \frac{AB}{AP} = \frac{AP}{PB}.$$

276. Preliminary Problem. *To divide a line-segment in extreme and mean ratio.*



Given the segment AB.

To divide AB in extreme and mean ratio, so that AB : AP = AP : PB.

Construction. Draw the line $BX \perp AB$ at B. On BX lay off BO equal to $\frac{1}{2}$ AB. With O as center and OB as radius, draw a circle. Draw line AO to cut the circumference in points M and N. On AB lay off AP equal to AM. Then P is the point sought to divide the line internally in the required ratio.

Proof. Let s = the number of units in AB.
and m = the number of units in AP.
$\therefore$ $s + m$ = the number of units in AN. Why?
and $s - m$ = the number of units in PB.

Forming a proportion that we know

$$\frac{s+m}{s} = \frac{s}{m}.$$ Why? (Art. 228, Cor.)

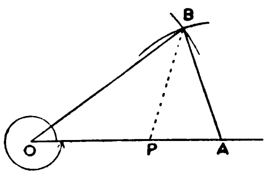
$\therefore \quad \dfrac{s + m - s}{s} = \dfrac{s - m}{m};$ Why?

$\therefore \quad \dfrac{m}{s} = \dfrac{s - m}{m};$

$\therefore \quad \dfrac{s}{m} = \dfrac{m}{s - m},$

or $\quad \dfrac{AB}{AP} = \dfrac{AP}{PB}.$

277. We are now ready to divide the perigon into five equal parts.



Given the perigon about O.

To divide it into five equal parts.

Construction. Draw the line-segment OA and divide it into extreme and mean ratio, so that $\dfrac{OA}{OP} = \dfrac{OP}{PA}$.

With A as a center and OP as a radius, construct an arc.

With O as a center and OA as a radius, construct an arc intersecting the first arc. Call the point of intersection B. Twice angle AOB is one-fifth of a perigon.

Proof. $\triangle$s ABO and ABP are similar. Why?

$\therefore$ $\triangle ABP$ is isosceles. Why?

$\angle APB = 2 \angle O$. Why?

$\therefore$ $\angle OBA = 2 \angle O$. Why?

$\therefore$ $\angle O$ = one-fifth of a straight angle. Why?

$\therefore$ $2 \angle O$ = one-fifth of a perigon. Why?

Corollary. *A perigon can be divided into* $5 \cdot 2^n$ *equal parts.*

278. Exercises.

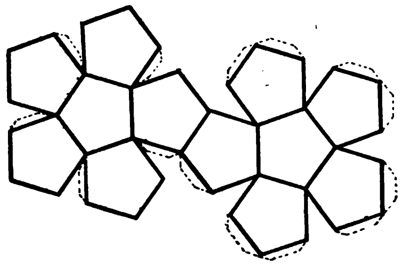
1. Solve the equation $\frac{s}{m} = \frac{m}{s - m}$ (Art. **277**) for m in terms of s, and show that $m = \frac{1}{2}(-1 \pm \sqrt{5})\,s$.

2. If AB or s is 4 inches long, find AP or m:

(a) By construction.

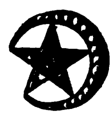
(b) By the formula of Exercise 1. Which sign before $\sqrt{5}$ is used?

3. Construct a regular pentagon.



4. Construct part of a pattern, as in the figure, for a regular dodecahedron. This is one of the five so-called *regular solids*, bounded by 12 faces which are regular and equal pentagons.

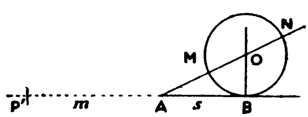
5. Construct the adjacent design.



Suggestion. Draw a circle 5 inches in diameter and another circle 4 inches in diameter internally tangent to the first at a point P. Mark the vertices of a regular pentagon on the inner circumference, starting at the point diametrically opposite to P. Join each vertex of this pentagon with the second following vertex, so forming the star.

279. External Division in Extreme and Mean Ratio.

Construction. In the figure, line AO cuts the circumference again at N. With AN as radius, and A as center, strike an arc cutting AB *produced* at P'.



We can then show that

$$AB : AP' = AP' : P'B \quad \text{or} \quad \frac{AB}{AP'} = \frac{AP'}{P'B}.$$

Point P' is then said to divide segment AB externally in extreme and mean ratio.

Proof. $AM : AB = AB : AN$,

or, $\frac{m - s}{s} = \frac{s}{m}$. Why?

$\therefore$ $\frac{m}{s} = \frac{m + s}{m}$. Why?

$\therefore$ $\frac{s}{m} = \frac{m}{m + s}$. Why?

$\therefore$ $\frac{AB}{AP'} = \frac{AP'}{P'B}$.

Note. In Exercise 1, page 248, two values of m are obtained by solving the given equation, which was derived in the problem of the internal division of a line-segment. If we call these values m_1 and m_2, we have

$$m_1 = \tfrac{1}{2}(-1 + \sqrt{5})\, s;$$
$$m_2 = -\tfrac{1}{2}(1 + \sqrt{5})\, s.$$

These will be of opposite signs, because $-1 + \sqrt{5}$ is positive, and $-1 - \sqrt{5}$ is negative. If we lay off m_1 to the right from A (figure on p. 246) we reach point P, the point of internal division. If we lay off the negative segment m_2 to the left from A, we would have a point P', the point of external division.

280. Exercises.

1. Solve $\frac{s}{m} = \frac{m}{m + s}$ for m and show that $m = \frac{1}{2}(1 \pm \sqrt{5})\, s$.

2. If $AB = 4$ inches, find AP':

   (a) By construction.

   (b) By the formula of Exercise 1. Which sign must be used?

3. If the two values of m obtained in Exercise 1 are denoted by m_1 and m_2 respectively, so that

$$m_1 = \tfrac{1}{2}(1 + \sqrt{5})\, s, \quad \text{and} \quad m_2 = \tfrac{1}{2}(1 - \sqrt{5})\, s,$$

show that m_2, laid off to the right from A, gives internal division.

281. Problem V. *To divide a perigon into fifteen equal parts.*

Analysis. $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$. Therefore by subtracting an angle which is one-fifth of a perigon from one that is one-third of a perigon and bisecting the remainder you will have an angle which is one-fifteenth of a perigon.

Make this construction.

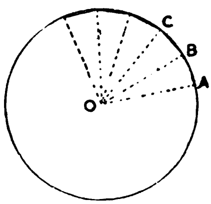
A perigon can be divided into $15 \cdot 2^n$ equal parts.

282. Problem VI. *To divide a circumference into equal parts.*

We have but to divide the perigon at the center into the required number of equal parts, and the arms of the angles will divide the circumference and also the circle into the required number of equal parts.

Therefore the division of a circumference into equal parts is limited to the number of equal parts into which we can divide the perigon.

283. Problem VII. *To inscribe a regular polygon in a circle.*



Given the circle whose center is O.

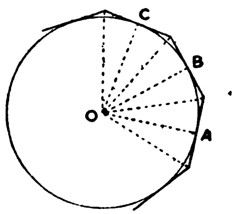
To inscribe a regular polygon of n sides.

Construction. Divide the circumference into n equal parts, the division points being A, B, C . . . join these points with straight lines. Polygon ABC . . . is the polygon required.

Prove this by drawing radii as shown in the figures, and proving congruent triangles.

Note. A regular polygon the number of whose sides is a prime number, such as 3, 5, 7, 11, 13, 17, ... can be constructed by rule and compass alone, if, and only if, the number of sides can be expressed by the formula $2^{2^n} + 1$, where n is an integer. When $n = 0, 1, 2, 3$, the number of sides is 3, 5, 17, 257; these are the only polygons with a prime number of sides not exceeding 257 which can be constructed with rule and compass alone.

284. Problem VIII. *To circumscribe a regular polygon about a circle.*



Given the circle whose center is *O*.

To circumscribe a regular polygon about it.

Construction. Divide the circumference into n equal parts and construct tangents at the points of division.

Prove by drawing in lines as shown in the figure, and proving congruent triangles.

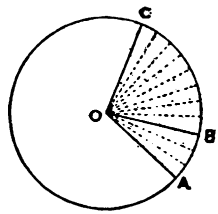
285. Problem IX. *To circumscribe a circle about a given regular polygon.*

Suggestion. Erect the perpendicular bisectors to two consecutive sides. The point where they meet is the center of the circumscribed circle. For proof see Art. **258.**

286. Problem X. *To inscribe a circle in a given polygon.*

Suggestion. Bisect two consecutive angles. The point of intersection of these bisectors is the center of the inscribed circle. For proof see Art. **258.**

287. Theorem XI. *In the same circle, or in equal circles, two sectors have the same ratio as their central angles.*



Given two sectors AOB **and** BOC.

To prove that sector AOB is to sector BOC as ∠ AOB is to ∠ BOC.

Prove this by dividing the angles into equal parts, and thus dividing the sectors into equal parts, and showing equal ratios.

This proof holds only for the commensurable case. For the incommensurable case see Art. **292**.

Corollary. *To find the area of a sector we find the ratio of its angle to a perigon, and take that portion of the area of the circle.*

In algebraic language, if α (alpha) is the number of radians in the angle, then

$$\frac{\alpha}{2\pi} = \frac{s}{\pi r^2};$$

from which $$s = \tfrac{1}{2} r^2 \alpha.$$

If the angle of the sector is d degrees, we shall have the proportion

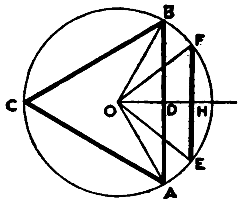
$$\frac{d}{360} = \frac{s}{\pi r^2}.$$

Then $$s = \pi r^2 \frac{d}{360}.$$

288. Exercises.

1. It is interesting to calculate the perimeters of several polygons inscribed in a given circle, using the trigonometric functions.

In the figure, ABC is an equilateral triangle, OD its apothem, BD its half-side. Let the radius of the circle be r. Let p denote the perimeter of the triangle.



Then $\qquad p = 3 \cdot 2\,BD.$

But $\qquad \dfrac{BD}{r} = \sin \angle\, DOB$

$$= \sin \frac{120^\circ}{2}.$$

$\therefore \qquad BD = r \sin 60^\circ.$

$\therefore \qquad p = 6\,r \sin 60^\circ.$

Let EF be one side of a regular inscribed polygon of n sides, OH its apothem, HF its half side.

Then $\qquad p = n \cdot 2\,HF = 2\,n\,HF.$

Now $\qquad \angle\, HOF = \dfrac{360^\circ}{2n}.$

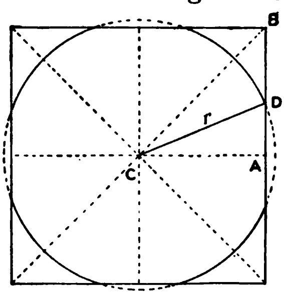
$\therefore \qquad HF = r \sin \dfrac{360^\circ}{2\,n}$

$$= r \sin \frac{180^\circ}{n}.$$

$\therefore \qquad p = 2\,nr \sin \dfrac{180^\circ}{n}.$

Calculate p when $n = 4, 6, 10, 30, 60$, using the table on page 205. Tabulate results. Note the value of the ratio of p to r. Compare with Art. **264**.

2. Explain the following construction for finding where to cut off the corners of a square board to make it a regular octagon.



Draw the dotted lines. Bisect angle ACB by the line CD. With CD as a radius draw a circle. This circle will mark the corners of a regular octagon.

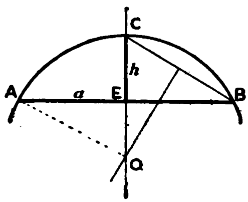
If the side of the square is 10 inches find the side of the octagon. (Apply Art. **215** to find AD.)

3. If s is the side of a regular octagon and a is its apothem, show that very nearly,

$$s : a = 29 : 40.$$

If $a = 10$ inches, what is the length of s, using this proportion? Check by construction.

4. To draw the arc of a circular segment of given base AB, and height EC.



Suggestions. From similar triangles,

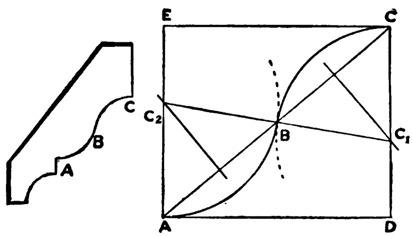
$$\frac{a}{h} = \frac{r}{m}. \quad (2m = BC.)$$

$$r = \frac{a}{h} m = \frac{1}{2} \cdot \frac{a}{h} \sqrt{a^2 + h^2}.$$

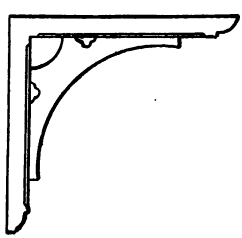
Give reasons for these statements.

Assuming values for a and h make the construction, then solve for the value of r by algebra, and compare results.

5. Construct design for moulding like that in the illustration. Here the curve ABC consists of two arcs each less than a quadrant. To construct it draw rectangle $ADCE$, making $AD = 3.5$ inches and $CD = 3$ inches. Point B is the center of this rectangle and AC is its diagonal. Locate C_1 by drawing the perpendicular bisector of BC. Draw arc BC with C_1 as center and C_1C as radius.



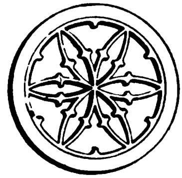
By changing the dimensions of the rectangle, you can get other curves.



6. Construct a design for a bracket.

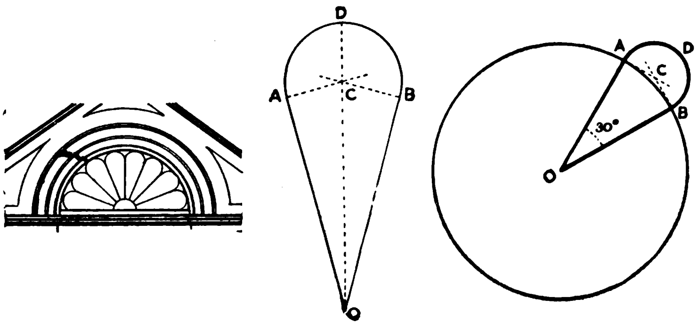
Follow the adjacent design, or construct one of your own.

7. Construct the principal outlines of the design in the adjacent figure.



If the radius of the circle which contains the "star" is 3 inches, calculate the perimeter of the star.

8. Construct a part of a rosette of 12 leaves so that $OA = OB = 3$ inches.



Explain the construction.

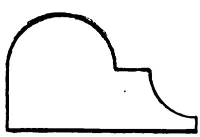
9. Construct a rosette of six leaves which shall just fill a circle of 3 inches radius. Calculate the length of AC and OC (see figure above) in this case. $\angle\ AOB$ is now 60°.

If $OA = r$, and if there are six leaves, show that

$$AC = \tfrac{1}{3}\, r\sqrt{3}, \text{ and } OC = \tfrac{2}{3}\, r\sqrt{3}.$$

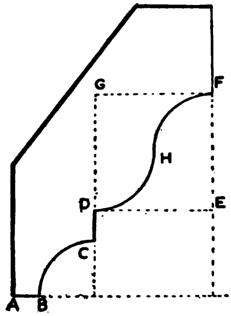
10. Construct the principal figures forming this design for linoleum, making the lines of your drawing about 6 times as long as those of the illustration.

11. Construct a moulding design similar to that in the figure and 4 inches wide.

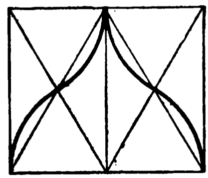


Calculate the area of your figure.

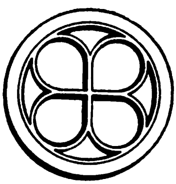
12. Construct a design for a moulding like that in the illustration. Make AB and CD each half an inch long, and use a radius of one inch for arcs BC, DH and HF, each of which is a quarter of a circumference.

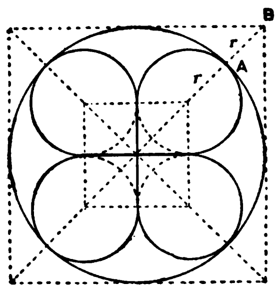


13. Construct this arch so that your drawing shall be 4 inches wide. Calculate the height and the area of the arch when the width of the base is 10 feet.



14. Construct a rosette of four leaves like that in the illustration. If R is the radius of the large circle, show that the radius of the small circles is

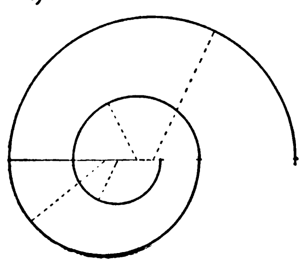




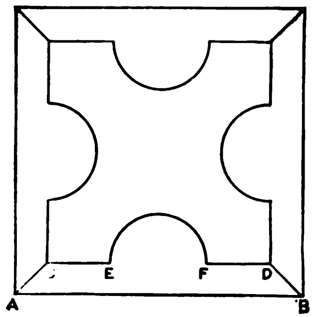
$$r = R(\sqrt{2} - 1).$$

Show that $r = AB$.

15. Construct a spiral design using semicircumferences joined together at their extremities.



16. Construct this figure, taking $AB = 4.5$ inches, $CD = 3.5$ inches, $EF = 1.5$ inches. Calculate the perimeter and the area of the four-pointed figure.

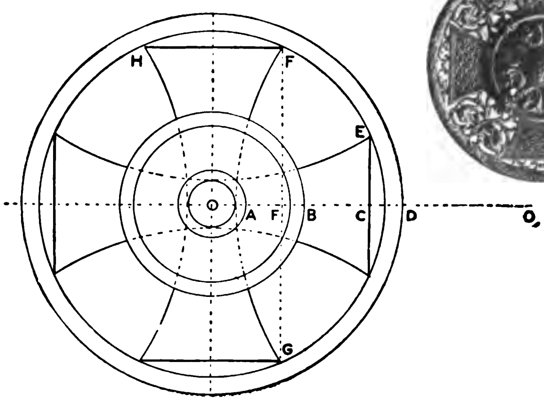


17. Construct the figure below, using the following dimensions or one-half of these dimensions:

$$OD = 2\tfrac{1}{2} \text{ inches}, OE = 2\tfrac{1}{4} \text{ inches.}$$
$$OB = 1\tfrac{1}{2} \text{ inches}, OA = \tfrac{1}{2} \text{ inch.}$$
$$OO_1 = 5 \text{ inches}, O_1F = 5\tfrac{5}{8} \text{ inches} = O_1G.$$

Calculate OF_1 and hence HF, and check by drawing.

Calculate FG, and check by drawing.

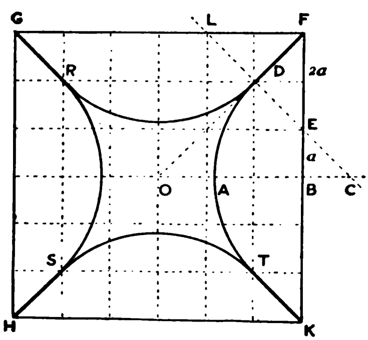


18. Construct the principal outline of the figure below.

19. Construct the principal lines of the figure below.

20. Construct a design similar to the illustration and four times as large.

21. Suggestions for the construction of the following design. *FGHK* is a square whose side is $6a$. Line *CD* is drawn so as to cut off the corner of the square, making $EF = FL = 2a$. With *C* as a center and *CD* as a radius construct arc *AD*. Similarly construct the other arcs. Construct such a figure making $a = \frac{1}{2}$ inch.



(1) Show that the two arcs meeting at each corner are tangent to each other.

(2) Show that $CD = 2\sqrt{2}\,a$.

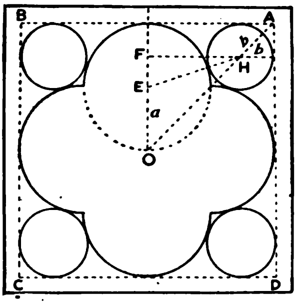
(3) Show that area of sector $ACD = \pi a^2$.

(4) Show that area of triangle $OCD = 4a^2$.

(5) Show that area $DRST$ = 8 times $OAD = 8(4-\pi)a^2$.

(6) Show that perimeter $DRST = 4\pi\sqrt{2}\,a$.

22. Construct this quadrifoil design.



Side of dotted square $= 4a$.
Radius of large circle $= a$.
Let the radius of small circle $= b$.
Prove the following:

(1) Area of quadrifoil $= 2a^2(\pi + 2)$.

(2) Perimeter of quadrifoil $= 4\pi a$.

(3) In the triangle EHF, $EH = a + b$,
$FH = 2a - b$,
$EF = a - b$.

Then by the Pythagorean theorem,

$$(a+b)^2 = (2a-b)^2 + (a-b)^2.$$

Expanding and solving for b, show that

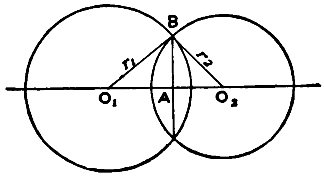
$$b = a(4 - 2\sqrt{3}).$$

(4) Now calculate OH, from triangle OHF;

$$OH = 2a\sqrt{2}(\sqrt{3} - 1).$$

Draw a figure, using $a = 1$ inch, and check your calculation.

23. Two circles intersect; radii are given r_1 and r_2; distance between the centers $O_1O_2 = d$. Calculate O_1A and AB.



Suggestion: Let $O_1A = x$ and $AB = y$.

Then $AO_2 = d - x$.

From triangle O_1AB; $\overline{AB}^2 = r_1^2 - x^2$.

From triangle AO_2B; $\overline{AB}^2 = r_2^2 - (d - x)^2$.

Therefore
$$\begin{aligned} r_1^2 - x^2 &= r_2^2 - (d - x)^2 \\ &= r_2^2 - d^2 + 2\,dx - x^2. \end{aligned}$$

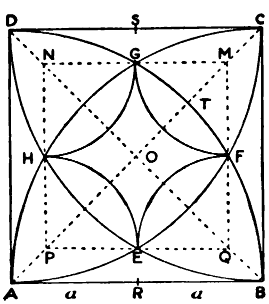
Therefore
$$x = \frac{r_1^2 - r_2^2 + d^2}{2\,d} = O_1A.$$

Then
$$AB = \sqrt{r_1^2 - \overline{O_1A}^2}.$$

Calculate O_1A and AB when $r_1 = 2$ inches, $r_2 = 1.5$ and $d = 2.5$ inches. Check by drawing.

24. Draw a square (four inches on a side) and within it construct the figure below.

A, B, C, D are the centers of the large arcs and M, N, P, Q are the centers of the small arcs.



If $AB = 2a$, show that:

(1) Area $BCGHA = \pi a^2$.

(2) Area $AEFCGH = 2a^2(\pi - 2)$.

(3) Length $MG = a(\sqrt{3} - 1)$.

Suggestion. Take R, S as mid-points of AB, CD.

Then in triangle BGR, $BG = 2a, BR = a$.

Therefore $RG = a\sqrt{3}$.

Then $GS = 2a - RG = a(2 - \sqrt{3})$.

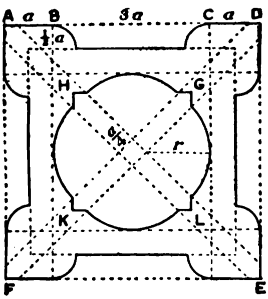
Then $MG = a - GS$.

(4) Area $MGF = \frac{1}{2}\pi a^2(2 - \sqrt{3})$.

(5) Area $GHEF = 2a^2(2 - \sqrt{3})(4 - \pi)$.

Suggestion. From square $MNPQ$ subtract four times area MGF.

25. (1) Construct the following figure, taking $a = 1$ inch.



(2) Calculate the perimeter and the area of the main figure whose corners are $AFED$.

Answers. Perimeter $= (16 + 2\pi)a$.

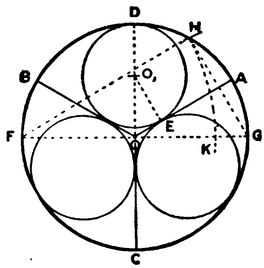
Area $= (19 + \frac{1}{2}\pi)a^2$.

(3) Calculate perimeter and area of figure $GHKL$.

Answers. Perimeter $= \dfrac{\pi a}{4}(12 - \sqrt{2}) + a\sqrt{2}$.

$$\text{Area} = \frac{\pi a^2}{32}(73 - 12\sqrt{2}) + \frac{a^2}{16}(25 - 2\sqrt{2}).$$

26. To inscribe three equal tangent circles in a given circle. Trisect the circumference of the given circle at points A, B, C. Draw OA, OB, OC. Bisect angle AOB by the line OD. The problem then is to draw a circle tangent to the given circle at D and tangent to OA and OB.



We can calculate the position of center O_1 as follows:

Let $OA = R$ and $OO_1 = 2x$.

Then in $\triangle OEO_1$, angle $E = 90°$,
angle $O = 60°$,
angle $O_1 = 30°$.

Therefore $OE = \frac{1}{2} OO_1 = x$.

Hence $EO_1 = \sqrt{\overline{OO_1}^2 - \overline{OE}^2} = x\sqrt{3}$.

But $O_1D = O_1E$.

Therefore $O_1D = x\sqrt{3}$.

Then we have

$$OO_1 + O_1D = R$$

or $$2x + x\sqrt{3} = R.$$

Therefore
$$x = \frac{R}{2+\sqrt{3}}$$
$$= R(2-\sqrt{3})$$
$$= 2R - R\sqrt{3}.$$

From this we can easily construct x.

For $FG = 2R$ and $FH = R\sqrt{3}$, if we make $GH = R$.

Therefore $KG = FG - FK = 2R - R\sqrt{3} = x$.

Now make $OO_1 = 2\ KG$, thus locating O_1. Using O_1D for a radius draw the circle.

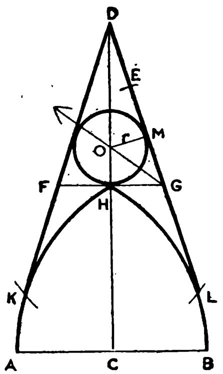
27. To inscribe three equal circles in a given circle, the radius of the small circle can be taken *approximately* from the proportion

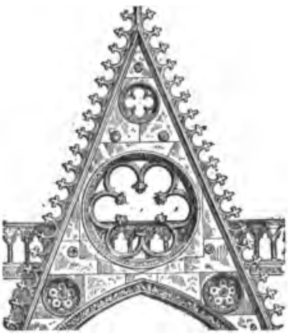
$$r : R = 13 : 28.$$

Find the value of r in terms of R in Exercise 26 and find the ratio of r to R. Determine to how many decimal places the ratio given in this exercise agrees with it.

If $R = 10$ inches, calculate r to two decimal places and check by construction.

28. Construct the design here shown, consisting of a Gothic arch, two tangent lines, and inscribed circle.





(a) Construct the arch, making $AB = 3$ inches.

(b) Draw tangent lines from point D, taking D so that $CH = HD$.

(c) Inscribe the circle in triangle FGD. O is found as the intersection of HD with the bisector of $\angle FGD$. Point E is used in drawing this bisector, GE being made equal to GF. With E and F as centers, and a radius more than half of EF, draw two intersecting arcs, and through their intersection and point G draw line GO.

If $AC = CB = a$, show that

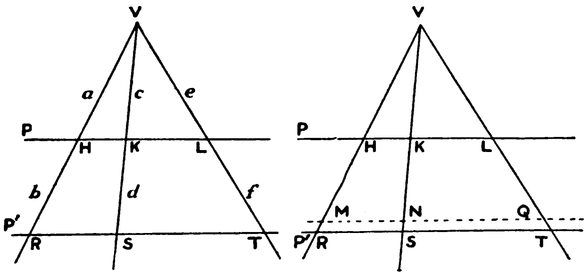
(1) $$CD = 2a\sqrt{3};$$

(2) $$\overline{DK}^2 = 9a^2, \text{ or } DK = 3a.$$

Suggestion. Regard DC produced as a secant of the circle to which DK is tangent, and apply Art. **228,** Corollary.

PART III.—INCOMMENSURABLE CASES.

289. Proof for Incommensurable Case of Theorem I, Chapter 6. *If a pencil of lines is cut by a pencil of parallels, the corresponding segments are in proportion.*



Given the pencil whose vertex is V cut by the parallels P and P', so that segments a and b are incommensurable.

To prove that $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

Analysis. Since segments a and b are incommensurable, a unit which will divide segment a exactly when laid off on segment b will leave a remainder. Suppose u is the length of such a unit, which, when laid off on segment b, will leave a remainder MR. A line drawn through M parallel to line P' will cut segment KN commensurable with segment c, and LQ commensurable with e on the other two rays respectively, leaving remainders NS and QT. Again, if we choose a unit u' less than MR, which is less than unit u, we shall have segments HM' commensurable with a, KN' commensurable with c, and LQ' commensurable with e, but with remainder in each case less than in the first case. And so by continually diminishing the unit of measure we can make the remainders as small as we please and make the segment HM approach b, KN approach d, LQ approach f, as limits

Proof. Since segment a and segment MH are commensurable

$$\frac{a}{HM} = \frac{c}{KN} = \frac{e}{LQ}. \quad \text{Art. 196.}$$

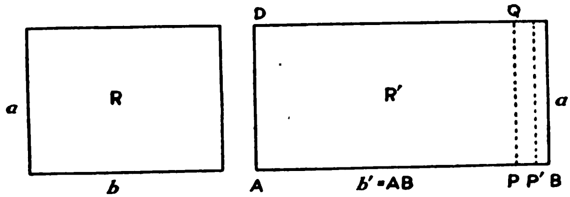
As we decrease the unit of measure,

$$HM \doteq b, \quad KN \doteq d, \quad LQ \doteq f.$$

Then $$\frac{a}{HM} \doteq \frac{a}{b}, \quad \frac{c}{KN} \doteq \frac{c}{d}, \quad \frac{e}{QL} \doteq \frac{e}{f}.$$

$\therefore$ $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f}. \quad \text{Art. 267.}$$

290. Proof for Incommensurable Case of Theorem I, Chapter 7. *Two rectangles having equal altitudes are to each other as their bases.*



Given the rectangles R and R' with equal altitudes a, and incommensurable bases b and b' respectively.

To prove: $\dfrac{\text{rectangle } R}{\text{rectangle } R'} = \dfrac{b}{b'}.$

Analysis. Since b and b' are incommensurable bases, a unit u which will divide base b exactly, when laid off on base b', will leave a remainder PB, and will give segment AP commensurable with base b. Choose a new unit less than PB; this will leave a remainder $P'B$ less than PB, making AP' commensurable with b. By making the unit of measure continually smaller we can make the remainder as small as we please, and thus make the segment AP approach base b' as its limit. By drawing perpendiculars at points P, we have the rectangles $APQD$ approaching rectangle R' as limit. So our proof consists in showing that by this process we have two variables constantly equal, and that the ratios of the required proportion are the limits of these equal variables.

Proof. Since b and AP are commensurable,

$$\frac{\square R}{\square AQ} = \frac{b}{AP}.$$

As we decrease the unit of measure,

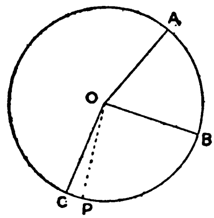
$$\square AQ \doteq \square R'; \qquad \text{base } AP \doteq \text{base } b'.$$

$$\therefore \quad \frac{\square R}{\square AQ} \doteq \frac{\square R}{\square R'}; \qquad \frac{b}{AP} \doteq \frac{b}{b'}.$$

$$\therefore \quad \frac{\square R}{\square R'} = \frac{b}{b'}. \quad \text{Art. 267.}$$

291. Proof for Incommensurable Case of Theorem XI, Chapter 7.

In the same circle, two sectors have the same ratio as their central angles.



Given the sectors AOB and BOC in the circle whose center is O, $\angle$s AOB and BOC being incommensurable.

To prove: $\dfrac{\textit{sector } AOB}{\textit{sector } BOC} = \dfrac{\textit{angle } AOB}{\textit{angle } BOC}.$

Analysis. A unit of angle u, which will divide $\angle$ AOB exactly, when laid off on $\angle$ BOC will leave a remainder $\angle$ POC. Let the student complete this analysis, leading to the following proof.

Proof. Since ∠s AOB and BOP are commensurable,

$$\frac{\text{sector } AOB}{\text{sector } BOP} = \frac{\angle AOB}{\angle BOP}.$$

As we decrease the unit of measure,

$$\text{sector } BOP \doteq \text{sector } BOC; \quad \angle BOP \doteq \angle BOC.$$

$$\therefore \quad \frac{\text{sector } AOB}{\text{sector } BOP} \doteq \frac{\text{sector } AOB}{\text{sector } BOC}; \quad \frac{\angle AOB}{\angle BOP} \doteq \frac{\angle AOB}{\angle BOC}.$$

$$\therefore \quad \frac{\text{sector } AOB}{\text{sector } BOC} = \frac{\angle AOB}{\angle BOC}. \quad \text{Art. } \mathbf{267}.$$

292. Summary of Chapter 7.

Part I—Areas of Plane Figures.

Rectilinear figures.

Theorem I. Two rectangles having equal altitudes are to each other as their bases.

Corollary 1. Two rectangles having equal bases are to each other as their altitudes.

Corollary 2. Two parallelograms having equal altitudes are to each other as their bases, and having equal bases are to each other as their altitudes.

Corollary 3. Two triangles having equal altitudes are to each other as their bases and having equal bases are to each other as their altitudes.

Theorem II. Two rectangles are to each other as the product of (the numerical measure of) their bases and altitudes.

Theorem III. The area of a rectangle is equal to the product of its base by its altitude.

This means that the number which represents the square units in the area is equal to the product of the numbers which represent the linear units in the base and altitude.

Corollary 1. The area of a parallelogram equals the product of base by altitude.

Corollary 2. The area of a triangle equals half the product of base by altitude.

Corollary 3. The area of a trapezoid equals half the product of the sum of the bases by the altitude.

Theorem IV. Two similar triangles are to each other as the squares of any two corresponding sides, or as the squares of any two corresponding altitudes.

Theorem V. Two similar polygons are to each other as the squares of any two corresponding sides.

Problem I. To construct a polygon similar to a given polygon and having a given ratio to it.

Definition. Regular Polygons. Radii of Regular Polygons. Apothems.

Theorem VI. In a regular polygon the bisectors of the angles meet in a point equidistant from the sides, and the perpendicular bisectors of the sides meet in a point equidistant from the vertices.

Corollary. The radii of a regular polygon are equal. The apothems are equal.

Theorem VII. The area of a regular polygon is equal to half the product of its perimeter by its apothem.

Theorem VIII. Two regular polygons of the same number of sides are similar.

Corollary 1. The perimeters of regular polygons of the same number of sides are to each other as their apothems, as their radii, to their sides.

Corollary 2. The areas of regular polygons are to each other as the squares of their apothems, of their radii, of their sides.

Circles.

Definition of Inscribed and Circumscribed Polygons.

Theory of Limits.

Assumption I. The area of a circle is the limit approached by the area of a regular circumscribed polygon, or by the area of a regular inscribed polygon, when the process of doubling the number of sides is steadily continued.

Assumption II. The length of the circumference of a circle is the limit approached by the perimeter of a regular circumscribed polygon, or the perimeter of a regular inscribed polygon as the process of doubling the number of sides is steadily continued.

Assumption III. If while approaching their limits, two variables are constantly equal, their limits are equal.

Theorem IX. The ratio of the circumference of a circle to its diameter is constant.

Corollary 1. Letting π express the ratio $\frac{c}{d}$, we have $c = \pi d$.

Corollary 2. Since $c = \pi d$ and $d = 2r$, $c = 2\pi r$.

Corollary 3. Two circumferences are proportional to their radii.

Theorem X. The area of a circle is equal to half the product of circumference by radius.

Corollary 1. The area of a circle $= \pi r^2$.

Corollary 2. The areas of circles are to each other as the squares of their radii.

Part II—Division of a Perigon.

Problem II. To bisect a perigon.

Corollary. To divide a perigon into $2 \cdot 2n$ equal parts.

Problem III. To trisect a perigon.

Corollary. To divide a polygon into $3 \cdot 2n$ equal parts.

Problem IV. To divide a perigon into five equal parts.

Preliminary Problem. To divide a line in extreme and mean ratio.

Corollary. To divide a perigon into $5 \cdot 2n$ equal parts.

External Division in Extreme and Mean Ratio.

Problem V. To divide a perigon into fifteen equal parts.

Corollary. To divide a perigon into $15 \cdot 2n$ equal parts.

Problem VI. To divide a circumference into equal parts.

Problem VII. To inscribe a regular polygon in a given circle.

Problem VIII. To circumscribe a regular polygon about a given circle.

Problem IX. To circumscribe a circle about a given polygon.

Problem X. To inscribe a circle in a given polygon.

Theorem XI. In the same or equal circles sectors have the same ratio as their central angles.

Corollary. If s represents the number of square units in the area of a sector and a the number of radians in its angle, then

$$s = \tfrac{1}{2} a r^2.$$

If the angle of the sector in degrees is d, then

$$s = \pi r^2 \frac{d}{360}.$$

Part III—Incommensurable Cases.

1. Into what parts is this chapter divided?

2. Are there any theorems that have been proved without the use of preceding theorems?

3. State the theorems that depend directly upon the theorems of Chapter II. State those that depend directly on theorems proved in each of the successive chapters.

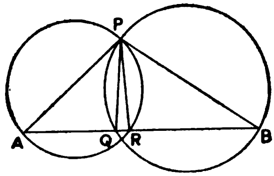
4. State the formula for the area of a circle. Then state the theorem from which it is derived. Then state the theorems that are immediately necessary for the proof of this theorem, then the theorems that are immediately necessary for the proof of these theorems and so on until you come to theorems that do not depend upon other theorems for their proof.

It would be well to indicate this chain of theorems by a diagram showing their connection, after the order of a family tree.

APPENDIX. SOME GEOMETRICAL PARADOXES.

The need for accurately drawn figures and careful proofs will be emphasized by a study of the following paradoxes. Read each one carefully; if you see nothing wrong, draw an accurate figure of your own; this will explain the fallacy in the first four cases.

(A). From a point not on a line two perpendiculars can be drawn to the line.

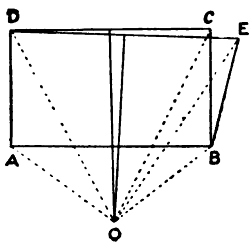


Draw two circles intersecting in P, and draw the diameters PA and PB. Draw AB meeting the circles in Q and R respectively.

Then $\angle PRA = 90°$ and $\angle PQB = 90°$. (Inscribed in semicircles.)

$\therefore$ $PQ \perp AB$ and $PR \perp AB$.

(B). To show that a right angle equals an obtuse angle.



Draw $\square ABCD$.

Draw $BE = BC$, forming obtuse $\angle ABE$. Draw DE.

Draw $\perp$ bisectors of CD and ED, and produce them to meet at O.

Draw OA, OB, OC, OD, OE.

Now prove $\triangle OBC \cong \triangle OBE$ as follows:

(1) Side OB is common.

(2) $BC = BE$, by construction.

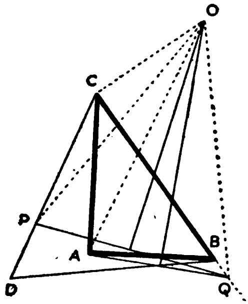
(3) $OC = OE$, because each is equal to OD. (Art. **35.**)

$\therefore$ $\triangle OBC \cong \triangle OBE$.

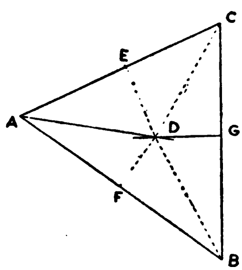
$\therefore$ $\angle OBC = \angle OBE$.

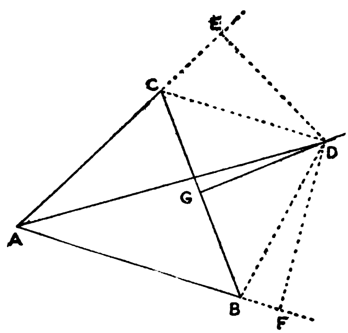
Subtracting $\angle OBA$ from each of these; $\angle ABC = \angle ABE$.

(C). **To prove part of an angle equal to the whole angle.**



Draw rt. △ ABC, with ∡ B and C less than 60°.
On BC, construct equilateral △ BCD.
On CD lay off $CP = CA$.
Let X = mid-point of AB.
Draw PX and produce it to meet CB at Q. Draw AQ.
Draw ⊥ bisectors of AQ and PQ, and produce them to meet at O.
Draw OC, OA, OP, OQ.
Then $OQ = OA = OP$. Also $CA = CP$ and $CO = CO$.
∴ △ $AOC \cong$ △ POC. ∴ $\angle ACO = \angle PCO$.
But $\angle ACO$ is part of $\angle PCO$.
This paradox is also involved in (B).
(D). To prove that every triangle is isosceles.





Given △ ABC. To prove $AB = AC$.
Draw bisector of $\angle A$ and the ⊥ bisector of side BC; produce these lines to meet at D.

Case I. Suppose D to fall within the triangle.

Draw DC and DB.

Draw $DE \perp AC$ and $DF \perp AB$.

$\triangle ADE \cong \triangle ADF$; $\therefore AE = AF$.

$\triangle EDC \cong \triangle FDB$, since $ED = FD$ and $CD = BD$. $\therefore EC = FB$.

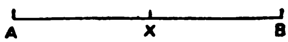
$\therefore AE + EC = AF + FB$, or, $AC = AB$.

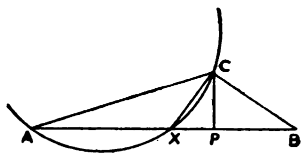
Case II. Suppose D to fall without the triangle.

Making a construction similar to that in Case I and using congruent triangles, prove $AE = AF$ and $CE = BF$.

$\therefore \quad AE - CE = AF - BF$, or, $AB = AC$.

(E). To prove part of a line-segment equal to the whole segment.





Given segment AB, and on it a point X between A and B.

To prove $AX = AB$.

On AX as chord draw a circle, and from B draw BC making $\angle ABC = \angle XCA$. (Notice that $\angle XCA$ is an inscribed angle and will be the same wherever BC meets the circle.)

Draw $CP \perp AB$.

$\triangle ABC$ is similar to $\triangle ACX$. (Equiangular.)

$\therefore \quad \triangle ABC : \triangle ACX = \overline{BC}^2 : \overline{CX}^2$. (Art. **255.**)

Also $\quad \triangle ABC : \triangle ACX = AB : AX$. (Common altitude.)

$\therefore \quad \overline{BC}^2 : AB = \overline{CX}^2 : AX$.

But $\quad \overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AB \cdot AP$; (Art. **119.**)

and $\quad \overline{CX}^2 = \overline{AC}^2 + \overline{AX}^2 - 2\,AX \cdot AP$.

$$\therefore \quad \frac{\overline{AC}^2 + \overline{AB}^2 - 2\,AB \cdot AP}{AB} = \frac{\overline{AC}^2 + \overline{AX}^2 - 2\,AX \cdot AP}{AX}.$$

$$\therefore \quad \frac{\overline{AC}^2}{AB} + AB - 2\,AP = \frac{\overline{AC}^2}{AX} + AX - 2\,AP.$$

$$\therefore \quad \frac{\overline{AC}^2}{AB} + AB = \frac{\overline{AC}^2}{AX} + AX.$$

$$\therefore \quad \frac{\overline{AC}^2}{AB} - AX = \frac{\overline{AC}^2}{AX} - AB.$$

$$\therefore \quad \frac{\overline{AC}^2 - AB \cdot AX}{AB} = \frac{\overline{AC}^2 - AB \cdot AX}{AX}$$

The numerators of these fractions are equal.

$\therefore \qquad AB = AX.$

Here the figure is correctly drawn and all the reasoning is correct, except the last step.

If the numerators of two equal fractions are equal we can infer that the denominators are equal, *except when both numerators are zero.* Two fractions like $\frac{0}{a}$ and $\frac{0}{b}$ are equal because both are equal to zero, regardless of whether a equals b or not. Now in our fractions above

$$\overline{AC}^2 - AB \cdot AX = 0,$$

because from similar $\triangle$ ABC and AXC,

$$AX : AC = AC : AB, \text{ or } \overline{AC}^2 = AX \cdot AB.$$

Therefore the equality of our two fractions tells us nothing about AB and AX.

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